# $\mathbb{L}^p$ Solutions of Backward Stochastic Differential Equations with Jumps

Song Yao*

**Abstract**

In this paper, we study a multi-dimensional backward stochastic differential equation with jumps (BSDEJ) that has non-Lipschitz generator and unbounded random time horizon. For any $p \in (1,\infty)$, we show that the BSDEJ with a $p$-integrable terminal condition admits a unique $\mathbb{L}^p$-type solution.

**Keywords:** Backward stochastic differential equation with jumps, $\mathbb{L}^p$ solution, non-Lipschitz generator.

## 1 Introduction

The backward stochastic equation(BSDE) was initiated by Bismut [1973] and later developed by Pardoux and Peng [1990] to a fully nonlinear version. It has since grown rapidly in theory and been applied to various areas, such as mathematical finance, stochastic optimal control, stochastic differential games and etc (see the references in El Karoui et al. [1997] or in Cvitanić et al. [1998]).

Tang and Li [1994] added into the BSDE a jump term that is driven by a Poisson random measure independent of the Brownian motion. (Practically speaking, for example, if the Brownian motion stands for the noise from the financial market, then the Poisson random measure can be interpreted as the randomness of the insurance claims.) The authors obtained the existence and uniqueness of a solution to such a BSDEJ when the terminal condition is square integrable and the generator is Lipschitz continuous in variables $y, z$ and $u$. Then Pardoux [1997] relaxed the Lipschitz condition on variable $y$ by assuming a monotonicity condition as well as a linear growth condition on variable $y$ instead. Later, Rong [1997] and Yin and Mao [2008] even degenerated the monotonicity condition to a weaker version so as to remove the Lipschitz condition on variable $z$. The unbounded random time horizon was considered in both Pardoux [1997] and Yin and Mao [2008].

Among those efforts to generalize the theory of BSDEs, some were devoted to weakening the square integrability of the terminal condition. El Karoui et al. [1997] demonstrated that for any $p$-integrable terminal condition with $p \in (1,\infty)$, the BSDE with Lipschitz continuous generator admits a unique solution, which is also $p$-integrable. Then Briand and Carmona [2000] reduced the Lipschitz condition on variable $y$ by a strong monotonicity condition as well as polynomial growth condition on variable $y$. Later, Briand et al. [2003] found that the polynomial growth condition is not necessary if one uses the monotonicity condition similar to that of Pardoux [1997].

In the present paper, we analyze the BSDEJ with unbounded random time horizon and under a non-Lipschitz generator condition which is slightly more general than that of Yin and Mao [2008]. We show the existence and uniqueness of an $\mathbb{L}^p$ solution of the BSDEJ given a $p$-integrable terminal condition in two cases $p \in (1,2]$ and $p \in (2,\infty)$.

The layout of this paper is simple. Section 2 deals with case $p \in (1,2]$. Given the infinite time horizon, we start by estimating the $\mathbb{L}^p$ norm of any solution of the BSDEJ in term of the $\mathbb{L}^p$ norm of the terminal condition and of the coefficients in the monotonicity condition, see Proposition 2.1. Next, we derive a stability-like result (Proposition 2.2), which claims that a sequence of solutions of BSDEJs is a Cauchy sequence in $\mathbb{L}^p$ norm when the sequence of their terminal conditions is so and when the solutions satisfy an asymptotic monotonicity condition. Then the

*Department of Mathematics, University of Michigan, Ann Arbor, MI 48109; email: songyao@umich.edu.

uniqueness directly follows, see Theorem 2.1. For the existence, we first show the existence for a bounded terminal condition by applying a method from Rong [1997] that approximates the non-Lipschitz generator by a sequence of Lipschitz generators via convolution smoothing, see Proposition 2.3. As to a $p$-integrable terminal condition, we truncate it as a bounded one and use Proposition 2.3 together with Proposition 2.2 to obtain the general existence result. Eventually, the infinite time horizon can easily been replaced by any unbound random one, see Corollary 2.1. In section 3, we strength the monotonicity condition in order to apply the conclusion of section 2 to get the existence and uniqueness result for case $p \in (2,\infty)$, see Theorem 3.1 and Corollary 3.1.

## 1.1 Notation and Preliminaries

Throughout this paper we consider a complete probability space $(\Omega,\mathcal{F},P)$ on which a $d$-dimensional Brownian motion $B$ is defined. Given a measurable space $(\mathcal{X},\mathscr{B}(\mathcal{X}))$, let $\mathfrak{p}$ be an $\mathcal{X}$-valued Poisson point process on $(\Omega,\mathcal{F},P)$ that is independent of $B$. Recall that the counting measure $N_{\mathfrak{p}}(dt,dx)$ of $\mathfrak{p}$ on $[0,\infty)\times\mathcal{X}$ has the compensator $E\big[N_{\mathfrak{p}}(dt,dx)\big]=\nu(dx)\,dt$ for some $\sigma$-finite measure $\nu$ on $\big(\mathcal{X},\mathscr{B}(\mathcal{X})\big)$. For any $t\in[0,\infty)$, we define $\sigma$-fields

$$\mathcal{F}^{\mathfrak{p}}_t \stackrel{\triangle}{=} \sigma\big(\mathfrak{p}_s, s\le t\big), \qquad \mathcal{F}^{B}_t \stackrel{\triangle}{=} \sigma\big(B_s, s\le t\big)$$

and augment $\mathcal{F}_t \stackrel{\triangle}{=} \sigma\big(\mathcal{F}^{\mathfrak{p}}_t\cup\mathcal{F}^{B}_t\big)$ by all $P$-null sets of $\mathcal{F}$. Clearly, the filtration $\mathbf{F}=\{\mathcal{F}_t\}_{t\in[0,\infty)}$ satisfies the *usual hypotheses* (cf. e.g., Protter [1990]). Let $\mathscr{P}$ denote the $\mathbf{F}$-progressively measurable $\sigma$-field on $[0,\infty)\times\Omega$. In addition, we set $\mathcal{F}_\infty \stackrel{\triangle}{=} \sigma\Big(\underset{t\in[0,\infty)}{\cup}\mathcal{F}_t\Big)\subset\mathcal{F}$ and $\mathbf{F}_\infty \stackrel{\triangle}{=} \mathbf{F}\cup\{\mathcal{F}_\infty\}=\{\mathcal{F}_t\}_{t\in[0,\infty]}$.

Let $\mathbb{H}$ denote a generic real Hilbert space with inner product $(\cdot,\cdot)_{\mathbb{H}}$ and the induced norm $|x|_{\mathbb{H}} \stackrel{\triangle}{=} \sqrt{(x,x)_{\mathbb{H}}}$, $\forall\, x\in\mathbb{H}$. For any $r\in(0,\infty)$, we define the following two functions on $\mathbb{H}$: [1]

$$\mathscr{D}(x) \stackrel{\triangle}{=} \mathbf{1}_{\{x\neq 0\}}\frac{1}{|x|_{\mathbb{H}}}x \quad\text{and}\quad \pi_r(x) \stackrel{\triangle}{=} \frac{r}{r\vee|x|_{\mathbb{H}}}x, \qquad \forall\, x\in\mathbb{H}.$$

Given $l\in\mathbb{N}$, the following spaces of functions will be used in the sequel:

1) Let $\mathbb{L}^1_+[0,\infty)$ be the space of all functions $\psi:[0,\infty)\mapsto[0,\infty)$ with $\int_0^\infty\psi(t)dt<\infty$, and let $\mathbb{L}^2_+[0,\infty)$ be the space of all functions $\psi:[0,\infty)\mapsto[0,\infty)$ with $\int_0^\infty\psi^2(t)dt<\infty$.

2) Let $\mathbb{L}^2_\nu=\mathbb{L}^2(\mathcal{X},\mathscr{B}(\mathcal{X}),\nu;\mathbb{R}^l)$ be the space of all $\mathbb{R}^l$-valued, $\mathscr{B}(\mathcal{X})$-measurable functions $u$ with $\int_{\mathcal{X}}|u(x)|^2\nu(dx)<\infty$. Clearly, $\mathbb{L}^2_\nu$ is a real Hilbert space with the inner product

$$\big(u_1,u_2\big)_{\mathbb{L}^2_\nu} \stackrel{\triangle}{=} \int_{\mathcal{X}}\big\langle u_1(x),u_2(x)\big\rangle\nu(dx), \qquad \forall\, u_1,u_2\in\mathbb{L}^2_\nu\,.$$

For any $u\in\mathbb{L}^2_\nu$, its (induced) norm is $\|u\|_{\mathbb{L}^2_\nu} \stackrel{\triangle}{=} \Big\{\int_{\mathcal{X}}|u(x)|^2\nu(dx)\Big\}^{\frac{1}{2}}$.

3) For any sub-$\sigma$-field $\mathcal{G}$ of $\mathcal{F}$, let

- $\mathbb{L}^0_+(\mathcal{G})$ be the space of all real-valued non-negative $\mathcal{G}$-measurable random variables;
- $\mathbb{L}^p_+(\mathcal{G}) \stackrel{\triangle}{=} \Big\{\xi\in\mathbb{L}^0_+(\mathcal{G}):\|\xi\|_{\mathbb{L}^p_+(\mathcal{G})} \stackrel{\triangle}{=} \Big\{E\big[\xi^p\big]\Big\}^{\frac{1}{p}}<\infty\Big\}$ for all $p\in[1,\infty)$;
- $\mathbb{L}^\infty_+(\mathcal{G}) \stackrel{\triangle}{=} \Big\{\xi\in\mathbb{L}^0_+(\mathcal{G}):\|\xi\|_{\mathbb{L}^\infty_+(\mathcal{G})} \stackrel{\triangle}{=} \underset{\omega\in\Omega}{\operatorname{esssup}}\,\xi(\omega)<\infty\Big\}$;
- $\mathbb{L}^0(\mathcal{G})$ be the space of all $\mathbb{R}^l$-valued, $\mathcal{G}$-measurable random variables;
- $\mathbb{L}^p(\mathcal{G}) \stackrel{\triangle}{=} \Big\{\xi\in\mathbb{L}^0(\mathcal{G}):\|\xi\|_{\mathbb{L}^p(\mathcal{G})} \stackrel{\triangle}{=} \Big\{E\big[|\xi|^p\big]\Big\}^{\frac{1}{p}}<\infty\Big\}$ for all $p\in[1,\infty)$;
- $\mathbb{L}^\infty(\mathcal{G}) \stackrel{\triangle}{=} \Big\{\xi\in\mathbb{L}^0(\mathcal{G}):\|\xi\|_{\mathbb{L}^\infty(\mathcal{G})} \stackrel{\triangle}{=} \underset{\omega\in\Omega}{\operatorname{esssup}}|\xi(\omega)|<\infty\Big\}$.

[1] See Lemma A.5 and Lemma A.6 for properties of functions $\mathscr{D}$ and $\pi_r$.

4) Let $\mathbb{D}^\infty_{\mathbf{F}}$ be the space of all $\mathbb{R}^l$-valued, $\mathbf{F}$-adapted RCLL[2] processes $X$ with

$$\|X\|_{\mathbb{D}^\infty_{\mathbf{F}}} \stackrel{\triangle}{=} \underset{(t,\omega)\in[0,\infty)\times\Omega}{\operatorname{esssup}} |X_t(\omega)| = \underset{\omega\in\Omega}{\operatorname{esssup}}\Big(\sup_{t\in[0,\infty)}|X_t(\omega)|\Big) < \infty.$$

5) For any $p \in [1,\infty)$, we let

- $\mathbb{D}^p_{\mathbf{F}}$ be the space of all $\mathbb{R}^l$-valued, $\mathbf{F}$-adapted RCLL processes $X$ with $\|X\|_{\mathbb{D}^p_{\mathbf{F}}} \stackrel{\triangle}{=} \Big\{E\Big[\sup\limits_{t\in[0,\infty)}|X_t|^p\Big]\Big\}^{\frac{1}{p}} < \infty$;
- $\mathbb{M}^p_{\mathbf{F}}(\mathbb{H})$ be the space of all $\mathbb{H}$-valued, $\mathbf{F}$-predictably measurable processes $X$ with

$$\|X\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{H})} \stackrel{\triangle}{=} \bigg\{E\bigg[\Big(\int_0^\infty |X_t|^2_{\mathbb{H}}\,dt\Big)^{\frac{p}{2}}\bigg]\bigg\}^{\frac{1}{p}} < \infty;$$

- $\mathbb{S}^p_{\mathbf{F}} \stackrel{\triangle}{=} \mathbb{D}^p_{\mathbf{F}} \times \mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d}) \times \mathbb{M}^p_{\mathbf{F}}\big(\mathbb{L}^2_\nu\big)$.

In this paper, we use the convention $\inf\{\emptyset\} \stackrel{\triangle}{=} \infty$ and let $c_p$ denote a generic constant depending only on $p$ (in particular, $c_0$ stands for a generic constant depending on nothing), whose form may vary from line to line.

## 1.2 BSDEs with Jumps

A parameter pair $(\xi, f)$ consists of a random variable $\xi \in \mathbb{L}^0(\mathcal{F}_\infty)$ and a function $f : [0,\infty)\times\Omega\times\mathbb{R}^l\times\mathbb{R}^{l\times d}\times\mathbb{L}^2_\nu \mapsto \mathbb{R}^l$ such that $f$ is $\mathscr{P}\times\mathscr{B}\big(\mathbb{R}^l\big)\times\mathscr{B}\big(\mathbb{R}^{l\times d}\big)\times\mathscr{B}\big(\mathbb{L}^2_\nu\big)/\mathscr{B}(\mathbb{R}^l)$-measurable.

**Definition 1.1.** *Given a parameter pair $(\xi, f)$, a triplet $(Y,Z,U)$ is called a solution of the backward stochastic differential equation with jumps that has terminal condition $\xi$ and generator $f$ $\big($BSDEJ$(\xi,f)$ for short$\big)$ if the followings holds:*

*i) $Y$ is an $\mathbb{R}^l$-valued, $\mathbf{F}$-adapted RCLL process, $Z$ is an $\mathbb{R}^{l\times d}$-valued, $\mathbf{F}$-progressively measurable processes, and $U$ is an $\mathbb{L}^2_\nu$-valued, $\mathbf{F}$-progressively measurable processes such that*

$$\int_0^\infty \Big(\big|f\big(s,Y_s,Z_s,U_s\big)\big| + \big|Z_s\big|^2 + \big\|U_s\big\|^2_{\mathbb{L}^2_\nu}\Big)\,ds < \infty, \quad P\text{-a.s.}; \tag{1.1}$$

*ii) it holds $P$-a.s. that*

$$Y_t = \xi + \int_t^\infty f(s,Y_s,Z_s,U_s)ds - \int_t^\infty Z_s\,dB_s - \int_{(t,\infty)}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,\infty). \tag{1.2}$$

Here, $\widetilde{N}_p(ds,dx) \stackrel{\triangle}{=} N_{\mathfrak{p}}(ds,dx) - \nu(dx)\,ds$. One knows that $\Big\{\int_{(0,t]}\int_{\mathcal{X}}\Psi_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\Big\}_{t\in[0,\infty)}$ is a martingale for any $\mathbb{L}^2_\nu$-valued, $\mathbf{F}$-progressively measurable processes $\Psi$ with $E\Big[\int_0^\infty\big\|\Psi_s\big\|^2_{\mathbb{L}^2_\nu}\,ds\Big] < \infty$.

**Remark 1.1.** *1) The two stochastic integrals in* (1.2) *are well-posed. To see this, we set $M^Z_t \stackrel{\triangle}{=} \int_0^t Z_s dB_s$, $\forall\, t\in[0,\infty)$, and define $\mathbf{F}_\infty$-stopping times*

$$\tau_k \stackrel{\triangle}{=} \inf\bigg\{t\in[0,\infty) : \int_0^t \big|Z_s\big|^2 ds > k\bigg\}, \quad \forall\, k\in\mathbb{N}.$$

*For any $k\in\mathbb{N}$, since $\Big\{M^Z_{t\wedge\tau_k}\Big\}_{t\in[0,\infty)}$ is a uniformly integrable martingale, there exists a $P$-null set $N_k$ such that*[3] *$\lim\limits_{t\to\infty} M^Z_{t\wedge\tau_k(\omega)}(\omega)$ exists for any $\omega\in N^c_k$. By* (1.1)*, one can find a $P$-null set $N_0$ such that for any $\omega\in N^c_0$, $\tau_{\mathfrak{k}}(\omega)=\infty$ for some $\mathfrak{k}=\mathfrak{k}(\omega)\in\mathbb{N}$. Hence, for any $\omega\in\bigcap\limits_{k\in\mathbb{N}\cup\{0\}} N^c_k$, $\lim\limits_{t\to\infty} M^Z_t(\omega) = \lim\limits_{t\to\infty} M^Z_{t\wedge\tau_{\mathfrak{k}}(\omega)}(\omega)$ exists. Put in another way, the limit*

$$\int_0^\infty Z_s\,dB_s \stackrel{\triangle}{=} \lim_{t\to\infty}\int_0^t Z_s\,dB_s \quad \text{exists } P\text{-a.s.}$$

[2]right-continuous, with limits from the left

[3]See, e.g., Theorem II.3.1 of Revuz and Yor [1999].

*Similarly, the limit*

$$\int_{(0,\infty)}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) \stackrel{\triangle}{=} \lim_{t\to\infty}\int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) \;\; \textit{exists} \;\; P\textit{-a.s.}$$

*2) Then one can deduce from* (1.2) *that*

$$Y_\infty \stackrel{\triangle}{=} \lim_{t\to\infty} Y_t = \xi, \quad P\textit{-a.s.} \tag{1.3}$$

We end the introduction by recalling Lemma 2.2 of Yin and SiTu [2003], an existence and uniqueness result of BSDEJs in Lipschitz case.

**Lemma 1.1.** *Let $(\xi, f)$ be a parameter pair such that*

*( i) $\xi \in \mathbb{L}^2(\mathcal{F}_\infty)$ and $E\Big[\big(\int_0^\infty |f(t,0,0,0)|dt\big)^2\Big] < \infty$;*

*(ii) For some $\phi_1 \in \mathbb{L}^1_+[0,\infty)$ and $\phi_2 \in \mathbb{L}^2_+[0,\infty)$, it holds $dt\otimes dP$-a.e. that*

$$\begin{aligned}&\big|f(t,\omega,y_1,z_1,u_1)-f(t,\omega,y_2,z_2,u_2)\big| \\ &\qquad \le \phi_1(t)|y_1-y_2|+\phi_2(t)\Big(|z_1-z_2|+\|u_1-u_2\|_{\mathbb{L}^2_\nu}\Big), \quad \forall\,(y_1,z_1,u_1),(y_2,z_2,u_2)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}\times\mathbb{L}^2_\nu .\end{aligned} \tag{1.4}$$

*Then the BSDEJ$(\xi,f)$ admits a unique solution $(Y,Z,U)\in\mathbb{S}^2_{\mathbf{F}}$.*

## 2 Case 1: $p\in(1,2]$

We start with an a priori estimate.

**Proposition 2.1.** *Let $(\xi,f)$ be a parameter pair with $\xi\in\mathbb{L}^p(\mathcal{F}_\infty)$. Suppose that $(Y,Z,U)$ is a solution of BSDEJ$(\xi,f)$ that satisfies*

$$\big\langle Y_t, f(t,Y_t,Z_t,U_t)\big\rangle \le \mathfrak{f}_t|Y_t| + a_t|Y_t|^2 + \ell_p\Big(|Z_t|^2+\big\|U_t\big\|^2_{\mathbb{L}^2_\nu}\Big), \quad dt\otimes dP\textit{-a.e.} \tag{2.1}$$

*for two non-negative $\mathbf{F}$-progressively measurable processes $\{\mathfrak{f}_t\}_{t\in[0,\infty)}$, $\{a_t\}_{t\in[0,\infty)}$ and for some constant $\ell_p\in\big[0,\frac{p-1}{2}\big)$. Set $A_t\stackrel{\triangle}{=}\int_0^t a_s ds$, $t\in[0,\infty)$. If*

$$A_\infty<\infty,\; P\textit{-a.s.} \quad \textit{and} \quad E\left[\sup_{t\in[0,\infty)}\big(e^{A_t}|Y_t|\big)^p\right]<\infty, \tag{2.2}$$

*then there exists a constant $c_{p,l}$ depending only on $p$ and $\ell_p$ such that*

$$\begin{aligned}&E\Big[\sup_{s\in[0,\infty)}\big(e^{A_s}|Y_s|\big)^p\Big]+E\left[\Big(\int_0^\infty e^{2A_s}|Z_s|^2ds\Big)^{\frac{p}{2}}+\Big(\int_0^\infty e^{2A_s}\big\|U_s\big\|^2_{\mathbb{L}^2_\nu}ds\Big)^{\frac{p}{2}}\right]\\ &\qquad\le\quad c_{p,l}\,E\left[e^{pA_\infty}|\xi|^p+\Big(\int_0^\infty e^{A_s}\mathfrak{f}_s ds\Big)^p\right].\end{aligned} \tag{2.3}$$

**Proof:** For any $k\in\mathbb{N}$, we define $\mathbf{F}_\infty$-stopping times

$$\tau_k\stackrel{\triangle}{=}\inf\left\{t\in[0,\infty):A_t+\int_0^t\Big(|f(s,Y_s,Z_s,U_s)|+|Z_s|^2+\big\|U_s\big\|^2_{\mathbb{L}^2_\nu}\Big)ds>k\right\}. \tag{2.4}$$

Given $\varepsilon\in(0,\infty)$, the function $\varphi_\varepsilon(t,x)\stackrel{\triangle}{=}\big(|x|^2+\varepsilon e^{-t}\big)^{\frac{1}{2}}$, $(t,x)\in[0,\infty)\times\mathbb{R}^l$ has the following derivatives of its $p$-th power:

$$\begin{aligned}&D_t\,\varphi^p_\varepsilon(t,x)=-\frac{p\,\varepsilon}{2}e^{-t}\varphi^{p-2}_\varepsilon(t,x), \quad D_i\,\varphi^p_\varepsilon(t,x)=p\,\varphi^{p-2}_\varepsilon(t,x)\,x^i, \quad \forall\, i\in\{1,\cdots,l\},\\ \text{and}\quad &D^2_{ij}\,\varphi^p_\varepsilon(t,x)=p\,\varphi^{p-2}_\varepsilon(t,x)\,\delta_{ij}+p(p-2)\varphi^{p-4}_\varepsilon(t,x)\,x^ix^j, \quad \forall\, i,j\in\{1,\cdots,l\}.\end{aligned} \tag{2.5}$$

Now fix $0 \le t < T < \infty$. For any $\varepsilon \in (0,\infty)$ and $k \in \mathbb{N}$, applying Itô's formula[4] to $e^{pA_s}\varphi_\varepsilon^p(s,Y_s)$ over the interval $[\,t\wedge\tau_k, T\wedge\tau_k]$ yields that

$$\begin{aligned}
& e^{pA_{t\wedge\tau_k}}\varphi_\varepsilon^p(t\wedge\tau_k, Y_{t\wedge\tau_k}) + \frac{1}{2}\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\,\mathrm{trace}\left(Z_s Z_s^T D^2\varphi_\varepsilon^p(s,Y_s)\right)ds \\
& + \sum_{s\in(t\wedge\tau_k, T\wedge\tau_k]} e^{pA_s}\Big\{\varphi_\varepsilon^p(s,Y_s) - \varphi_\varepsilon^p(s,Y_{s-}) - \big\langle D\varphi_\varepsilon^p(s,Y_{s-}), \Delta Y_s\big\rangle\Big\} \\
& \quad = \; e^{pA_{T\wedge\tau_k}}\varphi_\varepsilon^p(T\wedge\tau_k, Y_{T\wedge\tau_k}) + p\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\big[\varphi_\varepsilon^{p-2}(s,Y_s)\big\langle Y_s, f(s,Y_s,Z_s,U_s)\big\rangle - a_s\varphi_\varepsilon^p(s,Y_s)\big]ds \\
& \quad\quad + \frac{p\,\varepsilon}{2}\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s - s}\varphi_\varepsilon^{p-2}(s,Y_s)ds - p\left(M_T^{k,\varepsilon} - M_t^{k,\varepsilon} + \widetilde{M}_T^{k,\varepsilon} - \widetilde{M}_t^{k,\varepsilon}\right), \qquad P\text{-a.s.,}
\end{aligned} \tag{2.6}$$

where $M_r^{k,\varepsilon} \stackrel{\triangle}{=} \int_0^{r\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s)\langle Y_s, Z_s dB_s\rangle$ and $\widetilde{M}_r^{k,\varepsilon} \stackrel{\triangle}{=} \int_{(0,r\wedge\tau_k]}\int_{\mathcal{X}} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_{s-})\langle Y_{s-}, U_s(x)\rangle \widetilde{N}_{\mathfrak{p}}(ds,dx)$ for any $r \in [0,\infty)$. It follows from (2.5) that

$$\begin{aligned}
\mathrm{trace}\left(Z_s Z_s^T D^2\varphi_\varepsilon^p(s,Y_s)\right) \; &= \; p\,\varphi_\varepsilon^{p-2}(s,Y_s)|Z_s|^2 + p(p-2)\varphi_\varepsilon^{p-4}(s,Y_s)\cdot\sum_{i=1}^{l}\bigg(\sum_{j=1}^{d} Y_s^i Z_s^{ij}\bigg)^2 \\
&\ge \; p\,\varphi_\varepsilon^{p-2}(s,Y_s)|Z_s|^2 + p(p-2)\varphi_\varepsilon^{p-4}(s,Y_s)|Y_s|^2|Z_s|^2 \ge p(p-1)\varphi_\varepsilon^{p-2}(s,Y_s)|Z_s|^2.
\end{aligned} \tag{2.7}$$

On the other hand, Taylor's Expansion Theorem implies that

$$\begin{aligned}
& \sum_{s\in(t\wedge\tau_k, T\wedge\tau_k]} e^{pA_s}\Big\{\varphi_\varepsilon^p(s,Y_s) - \varphi_\varepsilon^p(s,Y_{s-}) - \big\langle D\varphi_\varepsilon^p(s,Y_{s-}), \Delta Y_s\big\rangle\Big\} \\
& \quad = \; p\sum_{s\in(t\wedge\tau_k, T\wedge\tau_k]} e^{pA_s}\int_0^1 (1-\alpha)\langle \Delta Y_s, D^2\varphi_\varepsilon^p\,(s,Y_s^\alpha)\,\Delta Y_s\rangle d\alpha \qquad \Big(\text{let } Y_s^\alpha \stackrel{\triangle}{=} Y_{s-} + \alpha\Delta Y_s\Big) \\
& \quad = \; p\sum_{s\in(t\wedge\tau_k, T\wedge\tau_k]} e^{pA_s}\int_0^1 (1-\alpha)\Big[\varphi_\varepsilon^{p-2}\,(s,Y_s^\alpha)\,|\Delta Y_s|^2 + (p-2)\varphi_\varepsilon^{p-4}\,(s,Y_s^\alpha)\,\big\langle \Delta Y_s, Y_s^\alpha\big\rangle^2\Big]d\alpha \\
& \quad \ge \; p(p-1)\sum_{s\in(t\wedge\tau_k, T\wedge\tau_k]} e^{pA_s}|\Delta Y_s|^2\int_0^1 (1-\alpha)\varphi_\varepsilon^{p-2}\,(s,Y_s^\alpha)\,d\alpha \\
& \quad \ge \; \frac{p}{2}(p-1)\sum_{s\in(t\wedge\tau_k, T\wedge\tau_k]} e^{pA_s}|\Delta Y_s|^2\left(|Y_{s-}|^2\vee|Y_s|^2 + \varepsilon e^{-s}\right)^{\frac{p}{2}-1} \\
& \quad \ge \; \frac{p}{2}(p-1)\int_{(t\wedge\tau_k, T\wedge\tau_k]}\int_{\mathcal{X}} e^{pA_s}\left(|Y_{s-}|^2\vee|Y_s|^2 + \varepsilon e^{-s}\right)^{\frac{p}{2}-1}|U_s(x)|^2 N_{\mathfrak{p}}(ds,dx).
\end{aligned} \tag{2.8}$$

In the last inequality we used the fact that $|Y_s^\alpha| = |(1-\alpha)Y_{s-} + \alpha Y_s| \le |Y_{s-}|\vee|Y_s|$. Since all processes in (2.6) are RCLL ones, plugging (2.7), (2.8) and (2.1) into (2.6) yields that $P$-a.s.

$$\begin{aligned}
& e^{pA_{t\wedge\tau_k}}\varphi_\varepsilon^p(t\wedge\tau_k, Y_{t\wedge\tau_k}) + p\left(\frac{p-1}{2} - \ell_p\right)\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\,\varphi_\varepsilon^{p-2}(s,Y_s)|Z_s|^2 ds \\
& + \frac{p}{2}(p-1)\int_{(t\wedge\tau_k, T\wedge\tau_k]}\int_{\mathcal{X}} e^{pA_s}\left(|Y_{s-}|^2\vee|Y_s|^2 + \varepsilon e^{-s}\right)^{\frac{p}{2}-1}|U_s(x)|^2 N_{\mathfrak{p}}(ds,dx) \\
& \quad \le \; \eta + p\,\ell_p\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s)\big\|U_s\big\|_{\mathbb{L}^2_\nu}^2 ds - p\left(M_T^{k,\varepsilon} - M_t^{k,\varepsilon} + \widetilde{M}_T^{k,\varepsilon} - \widetilde{M}_t^{k,\varepsilon}\right), \quad t\in[0,T],
\end{aligned} \tag{2.9}$$

where $\eta = \eta(k,\varepsilon) \stackrel{\triangle}{=} e^{pA_{T\wedge\tau_k}}\varphi_\varepsilon^p(T\wedge\tau_k, Y_{T\wedge\tau_k}) + p\int_0^{T\wedge\tau_k} e^{pA_s}\mathfrak{f}_s\varphi_\varepsilon^{p-1}(s,Y_s)ds + \frac{p\,\varepsilon}{2}\int_0^{T\wedge\tau_k} e^{pA_s - s}\varphi_\varepsilon^{p-2}(s,Y_s)ds$.

[4]see e.g. Ikeda and Watanabe [1981, Theorem II.5.1] or Protter [1990, Theorem II.32]

The Burkholder-Davis-Gundy inequality, (2.4), Lemma A.1 and (2.2) imply that

$$E\Big[\sup_{s\in[0,T]}|M^{k,\varepsilon}_s|^2+\sup_{s\in[0,T]}|\widetilde{M}^{k,\varepsilon}_s|^2\Big]\le c_0E\left[\int_0^{T\wedge\tau_k}e^{2pA_s}\left(\varphi_\varepsilon^{2p-4}(s,Y_s)|Y_s|^2|Z_s|^2+\varphi_\varepsilon^{2p-4}(s,Y_{s-})|Y_{s-}|^2\big\|U_s\big\|^2_{\mathbb{L}^2_\nu}\right)ds\right]$$
$$\le c_0ke^{2pk}E\left[\sup_{s\in[0,T\wedge\tau_k]}\varphi_\varepsilon^{2p-2}(s,Y_s)\right]\le c_0ke^{2pk}\left(E\left[\sup_{s\in[0,T\wedge\tau_k]}|Y_s|^{2p-2}\right]+\varepsilon^{p-1}\right)$$
$$\le c_0ke^{2pk}\left\{\left(E\left[\sup_{t\in[0,\infty)}\left(e^{A_t}|Y_t|\right)^p\right]\right)^{\frac{2p-2}{p}}+\varepsilon^{p-1}\right\}<\infty,$$

which implies that both $M^{k,\varepsilon}_{\cdot\wedge T}$ and $\widetilde{M}^{k,\varepsilon}_{\cdot\wedge T}$ are uniformly integrable martingales. As an RCLL process, $Y$ jumps countably many times along its $P$-a.s. paths, more precisely,

$$\big\{t\in[0,\infty):\Delta Y_t(\omega)=Y_t(\omega)-Y_{t-}(\omega)\neq 0\big\}\text{ is a countable set for }P\text{-a.s. }\omega\in\Omega. \tag{2.10}$$

Hence one can deduce that

$$E\int_{(0,T\wedge\tau_k]}\int_{\mathcal{X}}e^{pA_s}\left(|Y_{s-}|^2\vee|Y_s|^2+\varepsilon e^{-s}\right)^{\frac{p}{2}-1}|U_s(x)|^2N_{\mathfrak{p}}(ds,dx)$$
$$=E\int_0^{T\wedge\tau_k}e^{pA_s}\left(|Y_{s-}|^2\vee|Y_s|^2+\varepsilon e^{-s}\right)^{\frac{p}{2}-1}\int_{\mathcal{X}}|U_s(x)|^2\nu(dx)\,ds=E\int_0^{T\wedge\tau_k}e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s)\big\|U_s\big\|^2_{\mathbb{L}^2_\nu}ds. \tag{2.11}$$

Then letting $t=0$ and taking expectations in (2.9) yield that

$$E\int_0^{T\wedge\tau_k}e^{pA_s}\,\varphi_\varepsilon^{p-2}(s,Y_s)\left(|Z_s|^2+\big\|U_s\big\|^2_{\mathbb{L}^2_\nu}\right)ds\le\frac{1}{p\left(\frac{p-1}{2}-\ell_p\right)}E[\eta]. \tag{2.12}$$

Let $Y^{*,\varepsilon}_t\stackrel{\triangle}{=}\sup_{s\in[0,t]}e^{A_s}\varphi_\varepsilon(s,Y_s)$, $t\in[0,\infty)$. It follows from (2.9) that

$$E\Big[\big(Y^{*,\varepsilon}_{T\wedge\tau_k}\big)^p\Big]\le E[\eta]+p\,\ell_p\,E\int_0^{T\wedge\tau_k}e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s)\big\|U_s\big\|^2_{\mathbb{L}^2_\nu}ds+2p\,E\Big[\sup_{t\in[0,T]}|M^{k,\varepsilon}_t|+\sup_{t\in[0,T]}|\widetilde{M}^{k,\varepsilon}_t|\Big]. \tag{2.13}$$

The Burkholder-Davis-Gundy inequality implies that

$$2p\,E\Big[\sup_{t\in[0,T]}\big|M^{k,\varepsilon}_t\big|+\sup_{t\in[0,T]}\big|\widetilde{M}^{k,\varepsilon}_t\big|\Big]\le c_p\,E\left[\big[M^{k,\varepsilon},M^{k,\varepsilon}\big]_T^{\frac12}+\big[\widetilde{M}^{k,\varepsilon},\widetilde{M}^{k,\varepsilon}\big]_T^{\frac12}\right]$$
$$\le c_pE\left[\left(\int_0^{T\wedge\tau_k}e^{2pA_s}\varphi_\varepsilon^{2p-4}(s,Y_s)|Y_s|^2|Z_s|^2ds\right)^{\frac12}+\left(\int_{(0,T\wedge\tau_k]}\int_{\mathcal{X}}e^{2pA_s}\varphi_\varepsilon^{2p-4}(s,Y_{s-})|Y_{s-}|^2|U_s(x)|^2N_{\mathfrak{p}}(ds,dx)\right)^{\frac12}\right]$$
$$\le c_pE\left[\big(Y^{*,\varepsilon}_{T\wedge\tau_k}\big)^{\frac p2}\left\{\left(\int_0^{T\wedge\tau_k}e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s)|Z_s|^2ds\right)^{\frac12}+\left(\int_{(0,T\wedge\tau_k]}\int_{\mathcal{X}}e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_{s-})|U_s(x)|^2N_{\mathfrak{p}}(ds,dx)\right)^{\frac12}\right\}\right]$$
$$\le\frac12E\left[\big(Y^{*,\varepsilon}_{T\wedge\tau_k}\big)^p\right]+c_p\left\{E\int_0^{T\wedge\tau_k}e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s)|Z_s|^2ds+E\int_0^{T\wedge\tau_k}e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_{s-})\int_{\mathcal{X}}|U_s(x)|^2\nu(dx)\,ds\right\}$$
$$=\frac12E\left[\big(Y^{*,\varepsilon}_{T\wedge\tau_k}\big)^p\right]+c_pE\int_0^{T\wedge\tau_k}e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s)\left(|Z_s|^2+\big\|U_s\big\|^2_{\mathbb{L}^2_\nu}\right)ds, \tag{2.14}$$

where we used (2.10) in the last equality.

Now, let $c_{p,l}$ denote a generic constant depending only on $p$ and $\ell_p$, whose form may vary from line to line. Lemma A.1, (2.4) and (2.2) imply that

$$E\Big[\big(Y^{*,\varepsilon}_{T\wedge\tau_k}\big)^p\Big]\le E\left[\sup_{t\in[0,\infty)}\left(e^{A_t}|Y_t|\right)^p\right]+e^{pk}\varepsilon^{\frac p2}<\infty.$$

Then plugging (2.12) and (2.14) into (2.13), we can deduce from Young's Inequality and Lemma A.1 that

$$
\begin{aligned}
E\left[\left(Y^{*,\varepsilon}_{T\wedge\tau_k}\right)^p\right] \;&\le\; c_{p,l}\, E[\eta] \le c_{p,l}\, E\left[e^{pA_{T\wedge\tau_k}}\left(|Y_{T\wedge\tau_k}|^2+\varepsilon\right)^{\frac{p}{2}} + \left(Y^{*,\varepsilon}_{T\wedge\tau_k}\right)^{p-1}\int_0^{T\wedge\tau_k} e^{A_s}\mathfrak{f}_s ds + \varepsilon^{\frac{p}{2}}\int_0^{T\wedge\tau_k} e^{pA_s-\frac{ps}{2}}ds\right]\\
&\le\; c_{p,l}\, J + \frac{1}{2}E\Big[\left(Y^{*,\varepsilon}_{T\wedge\tau_k}\right)^p\Big],
\end{aligned}
\tag{2.15}
$$

where $J = J(k,\varepsilon) \triangleq E\Big[e^{pA_{T\wedge\tau_k}}|Y_{T\wedge\tau_k}|^p + \left(\int_0^\infty e^{A_s}\mathfrak{f}_s ds\right)^p\Big] + \varepsilon^{\frac{p}{2}}e^{pk}$. In the second inequality above we used the fact that $\varphi_\varepsilon^{p-2}(s,Y_s) \le \left(\varepsilon e^{-s}\right)^{\frac{p}{2}-1}$, $\forall\,(s,\omega)\in[0,\infty)\times\Omega$. Then it follows from (2.15) that

$$
E\Big[\sup_{s\in[0,T\wedge\tau_k]}\left(e^{A_s}|Y_s|\right)^p\Big] \le E\Big[\left(Y^{*,\varepsilon}_{T\wedge\tau_k}\right)^p\Big] \le c_{p,l}\, J, \tag{2.16}
$$

$$
\text{and that}\qquad E[\eta] \le c_{p,l}\, J + \frac{1}{2}E\Big[\left(Y^{*,\varepsilon}_{T\wedge\tau_k}\right)^p\Big] \le c_{p,l}\, J. \tag{2.17}
$$

In light of Young's Inequality, we can deduce from (2.12), (2.16) and (2.17) that

$$
\begin{aligned}
E\bigg\{\Big(\int_0^{T\wedge\tau_k} e^{2A_s}|Z_s|^2 ds\Big)^{\frac{p}{2}}\bigg\} \;&\le\; E\left\{\left(Y^{*,\varepsilon}_{T\wedge\tau_k}\right)^{\frac{p(2-p)}{2}}\left(\int_0^{T\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s)|Z_s|^2 ds\right)^{\frac{p}{2}}\right\}\\
&\le\; \frac{2-p}{2}E\Big[\left(Y^{*,\varepsilon}_{T\wedge\tau_k}\right)^p\Big] + \frac{p}{2}E\int_0^{T\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s)|Z_s|^2 ds\\
&\le\; \frac{2-p}{2}E\Big[\left(Y^{*,\varepsilon}_{T\wedge\tau_k}\right)^p\Big] + \frac{1}{p-1-2\ell_p}E[\eta] \le c_{p,l}\, J.
\end{aligned}
\tag{2.18}
$$

Similarly, we can deduce that $E\Big\{\left(\int_0^{T\wedge\tau_k} e^{2A_s}\big\|U_s\big\|^2_{\mathbb{L}^2_\nu}ds\right)^{\frac{p}{2}}\Big\} \le c_{p,l}\, J$. Summing it up with (2.16) and (2.18) as well as letting $\varepsilon\to 0$ yield that

$$
\begin{aligned}
&E\Big[\sup_{s\in[0,T\wedge\tau_k]}\left(e^{A_s}|Y_s|\right)^p\Big] + E\bigg[\Big(\int_0^{T\wedge\tau_k} e^{2A_s}|Z_s|^2 ds\Big)^{\frac{p}{2}} + \Big(\int_0^{T\wedge\tau_k} e^{2A_s}\big\|U_s\big\|^2_{\mathbb{L}^2_\nu}ds\Big)^{\frac{p}{2}}\bigg]\\
&\qquad\le\; c_{p,l}\, E\left[e^{pA_{T\wedge\tau_k}}|Y_{T\wedge\tau_k}|^p + \left(\int_0^\infty e^{A_s}\mathfrak{f}_s ds\right)^p\right].
\end{aligned}
\tag{2.19}
$$

We know from (2.1) and (2.2) that for $P$-a.s. $\omega\in\Omega$, $\tau_k(\omega)=\infty$ for some $k=k(\omega)\in\mathbb{N}$. It follows that

$$
\lim_{k\to\infty} Y_{T\wedge\tau_k} = Y_T, \quad P\text{-a.s.}, \tag{2.20}
$$

although the process $Y$ may not be left-continuous. Therefore, letting $k\to\infty$ and then letting $T\to\infty$ in (2.19), we can deduce (2.3) from the Monotone Convergence Theorem, the Dominated Convergence Theorem, (2.20), (1.3), and (2.2). □

In the rest of this section, we let $\theta:[0,\infty)\mapsto[0,\infty)$ be an increasing concave function with $\displaystyle\int_{0+}^1 \frac{1}{\theta(t)}dt = \infty$. Our goal of this section is the following existence and uniqueness result of BSDEJs for case "$p\in(1,2]$".

**Theorem 2.1.** *Let $(\xi,f)$ be a parameter pair such that $\xi\in\mathbb{L}^p(\mathcal{F}_\infty)$ and that for each $(t,\omega)\in[0,\infty)\times\Omega$,*

**(H1)** *the mapping $f(t,\omega,\cdot,\cdot,u)$ is continuous for any $u\in\mathbb{L}^2_\nu$.*

*Then the BSDEJ$(\xi,f)$ admits a unique solution $(Y,Z,U)\in\mathbb{S}^p_{\mathbf{F}}$ if the generator $f$ satisfies the following conditions for $dt\otimes dP$-a.e. $(t,\omega)\in[0,\infty)\times\Omega$:*

**(H2)** $|f(t,\omega,y,z,0)| \le (1+|y|)\beta_t + c(t)|z|$, $\forall\,(y,z)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}$;

**(H3)** $|f(t,\omega,y,z,u_1)-f(t,\omega,y,z,u_2)| \le c(t)\|u_1-u_2\|_{\mathbb{L}^2_\nu}$, $\forall\,(y,z,u_1,u_2)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}\times\mathbb{L}^2_\nu\times\mathbb{L}^2_\nu$;

**(H4)** $|y_1-y_2|^{p-1}\left\langle\mathscr{D}(y_1-y_2), f\big(t,\omega,y_1,z_1,u_1\big)-f\big(t,\omega,y_2,z_2,u_2\big)\right\rangle \le \lambda(t)\theta\big(|y_1-y_2|^p\big) + \Lambda_t|y_1-y_2|^p$
$+\widetilde{\Lambda}_t|y_1-y_2|^{p-1}\big(|z_1-z_2| + \|u_1-u_2\|_{\mathbb{L}^2_\nu}\big)$, $\quad\forall\,(y_1,z_1,u_1),(y_2,z_2,u_2)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}\times\mathbb{L}^2_\nu$;

*where*

*(h1)* $c(\cdot) \in \mathbb{L}^1_+[0,\infty) \cap \mathbb{L}^2_+[0,\infty)$ *and* $\lambda(\cdot) \in \mathbb{L}^1_+[0,\infty)$,

*(h2)* $\beta$, $\Lambda$ *and* $\widetilde{\Lambda}$ *are three non-negative* $\mathbf{F}$*-progressively measurable processes such that* $\Big\{ \int_0^\infty \beta_t dt,\ \int_0^\infty \Lambda_t dt,$ $\int_0^\infty \widetilde{\Lambda}_t^2 dt \Big\} \subset \mathbb{L}^\infty_+(\mathcal{F}_\infty)$, *and that* $E\int_0^\infty \widetilde{\Lambda}_t dt + E\int_0^\infty \big(\widetilde{\Lambda}_t\big)^{2+\varpi} dt < \infty$ *for some* $\varpi \in (0,\infty)$.

The proof of Theorem 2.1 relies on the following two results.

**Proposition 2.2.** *Let* $\big\{(\xi_n, f_n)\big\}_{n\in\mathbb{N}}$ *be parameter pairs such that* $\{\xi_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $\mathbb{L}^p(\mathcal{F}_\infty)$. *Assume that for any* $n \in \mathbb{N}$, *the BSDEJ*$(\xi_n, f_n)$ *has a solution* $(Y^n, Z^n, U^n) \in \mathbb{S}^p_{\mathbf{F}}$, *and that for any* $m, n \in \mathbb{N}$ *with* $m > n$, $(Y^{m,n}, Z^{m,n}, U^{m,n}) \stackrel{\triangle}{=} (Y^m - Y^n, Z^m - Z^n, U^m - U^n)$ *satisfies*

$$|Y^{m,n}_t|^{p-1} \Big\langle \mathscr{D}(Y^{m,n}_t), f_m\big(t, Y^m_t, Z^m_t, U^m_t\big) - f_n\big(t, Y^n_t, Z^n_t, U^n_t\big) \Big\rangle \\ \le \lambda(t)\theta\big(|Y^{m,n}_t|^p + \delta_n\big) + \Lambda_t |Y^{m,n}_t|^p + \widetilde{\Lambda}_t |Y^{m,n}_t|^{p-1}\left(|Z^{m,n}_t| + \|U^{m,n}_t\|_{\mathbb{L}^2_\nu}\right) + \eta^{m,n}_t, \quad dt\otimes dP\text{-a.e.} \tag{2.21}$$

*where*

*( i)* $\lambda(\cdot) \in \mathbb{L}^1_+[0,\infty)$, $\Lambda$ *and* $\widetilde{\Lambda}$ *are two non-negative* $\mathbf{F}$*-progressively measurable processes such that* $\Big\{ \int_0^\infty \Lambda_t dt,$ $\int_0^\infty \widetilde{\Lambda}_t^2 dt \Big\} \subset \mathbb{L}^\infty_+(\mathcal{F}_\infty)$,

*(ii)* $\delta_n \in \mathbb{L}^0(\mathcal{F}_\infty)$ *is a non-negative random variable, and* $\eta^{m,n}$ *is a non-negative process such that*

$$\lim_{n\to\infty} \sup_{m>n} E\left[\int_0^\infty \eta^{m,n}_t dt\right] = 0. \tag{2.22}$$

*If* $\lambda$ *is non-trivial* $\big($*i.e.* $\int_0^\infty \lambda(t)dt > 0\big)$, *we further assume that*

$$\sup_{n\in\mathbb{N}}\Big(\|Y^n\|_{\mathbb{D}^p_{\mathbf{F}}} + \|Z^n\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})} + \|U^n\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{L}^2_\nu)}\Big) + \sup_{n\in\mathbb{N}} E[\delta_n] < \infty \quad \text{and} \quad \lim_{n\to\infty} E[\delta_n] = 0. \tag{2.23}$$

*Then* $\big\{(Y^n, Z^n, U^n)\big\}_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $\mathbb{S}^p_{\mathbf{F}}$.

**Proof:** Let $a_t \stackrel{\triangle}{=} \Lambda_t + \frac{2}{p-1}\widetilde{\Lambda}_t^2$ and $A_t \stackrel{\triangle}{=} \int_0^t a_s ds$, $t \in [0,\infty)$. It easily follows from (h2) that $A_\infty \in \mathbb{L}^\infty_+(\mathcal{F}_\infty)$ with $\kappa_A \stackrel{\triangle}{=} \|A_\infty\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)} \le \big\|\int_0^\infty \Lambda_s\, ds\big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)} + \frac{2}{p-1}\Big\|\int_0^\infty \widetilde{\Lambda}_s^2\, ds\Big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}$. Fix $k, m, n \in \mathbb{N}$ with $m > n$. We define an $\mathbf{F}_\infty$-stopping time

$$\tau_k = \tau^{m,n}_k \stackrel{\triangle}{=} \inf\left\{ t \in [0,\infty) : \int_0^t \left(|Z^{m,n}_s|^2 + \big\|U^{m,n}_s\big\|^2_{\mathbb{L}^2_\nu}\right) ds > k \right\}. \tag{2.24}$$

Fix $0 \le t < T < \infty$. Similar to (2.6)-(2.8), applying Itô's formula to $e^{pA_s}\varphi^p_\varepsilon(s, Y^{m,n}_s)$ over the interval $[\,t\wedge\tau_k, T\wedge\tau_k]$ yields that

$$e^{pA_{t\wedge\tau_k}}\varphi^p_\varepsilon(t\wedge\tau_k, Y^{m,n}_{t\wedge\tau_k}) + \frac{p}{2}(p-1)\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\,\varphi^{p-2}_\varepsilon(s, Y^{m,n}_s)|Z^{m,n}_s|^2 ds \\ +\frac{p}{2}(p-1)\int_{(t\wedge\tau_k, T\wedge\tau_k]}\int_{\mathcal{X}} e^{pA_s}\left(|Y^{m,n}_{s-}|^2 \vee |Y^{m,n}_s|^2 + \varepsilon e^{-s}\right)^{\frac{p}{2}-1} |U^{m,n}_s(x)|^2 N_{\mathfrak{p}}(ds,dx) \\ \le e^{pA_{T\wedge\tau_k}}\varphi^p_\varepsilon(T\wedge\tau_k, Y^{m,n}_{T\wedge\tau_k}) + p\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\varphi^{p-2}_\varepsilon(s, Y^{m,n}_s)\big\langle Y^{m,n}_s, f_m(s, Y^m_s, Z^m_s, U^m_s) - f_n(s, Y^n_s, Z^n_s, U^n_s)\big\rangle ds \\ -p\int_{t\wedge\tau_k}^{T\wedge\tau_k} a_s e^{pA_s}\varphi^p_\varepsilon(s, Y^{m,n}_s)ds + \frac{p\,\varepsilon}{2}\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s - s}\varphi^{p-2}_\varepsilon(s, Y^{m,n}_s)ds \\ -p\left(M^{m,n,\varepsilon}_T - M^{m,n,\varepsilon}_t + \widetilde{M}^{m,n,\varepsilon}_T - \widetilde{M}^{m,n,\varepsilon}_t\right), \quad P\text{-a.s.,} \tag{2.25}$$

where

$$M_r^{m,n,\varepsilon} \triangleq \int_0^{r\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})\langle Y_s^{m,n}, Z_s^{m,n}dB_s\rangle$$
$$\text{and}\quad \widetilde{M}_r^{m,n,\varepsilon} \triangleq \int_{(0,r\wedge\tau_k]}\int_{\mathcal{X}} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_{s-}^{m,n})\big\langle Y_{s-}^{m,n}, U_s^{m,n}(x)\big\rangle \widetilde{N}_{\mathfrak{p}}(ds,dx)$$

for any $r \in [0,\infty)$. One can deduce from (2.21) that $dt\otimes dP$-a.e.

$$\begin{aligned}
&\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})\langle Y_s^{m,n}, f_m(s,Y_s^m,Z_s^m,U_s^m)-f_n(s,Y_s^n,Z_s^n,U_s^n)\rangle\\
&\quad= \left(\frac{|Y_s^{m,n}|}{\varphi_\varepsilon(s,Y_s^{m,n})}\right)^{2-p}|Y_s^{m,n}|^{p-1}\big\langle \mathscr{D}(Y_s^{m,n}), f_m(s,Y_s^m,Z_s^m,U_s^m)-f_n(s,Y_s^n,Z_s^n,U_s^n)\big\rangle\\
&\quad\le \lambda(s)\theta\big(|Y_s^{m,n}|^p+\delta_n\big)+\Lambda_s|Y_s^{m,n}|^p+\widetilde{\Lambda}_s|Y_s^{m,n}|^{p-1}\big(|Z_s^{m,n}|+\|U_s^{m,n}\|_{\mathbb{L}^2_\nu}\big)+\eta_s^{m,n}\\
&\quad\le \lambda(s)\theta\big(|Y_s^{m,n}|^p+\delta_n\big)+\Lambda_s\varphi_\varepsilon^p(s,Y_s^{m,n})+\widetilde{\Lambda}_s\varphi_\varepsilon^{p-1}(s,Y_s^{m,n})\big(|Z_s^{m,n}|+\|U_s^{m,n}\|_{\mathbb{L}^2_\nu}\big)+\eta_s^{m,n}\\
&\quad\le \lambda(s)\theta\big(|Y_s^{m,n}|^p+\delta_n\big)+a_s\varphi_\varepsilon^p(s,Y_s^{m,n})+\frac{p-1}{4}\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})\Big(|Z_s^{m,n}|^2+\big\|U_s^{m,n}\big\|^2_{\mathbb{L}^2_\nu}\Big)+\eta_s^{m,n}.
\end{aligned}\tag{2.26}$$

As $\varphi_\varepsilon^{p-2}(s,Y_s^{m,n}) \le \big(\varepsilon e^{-s}\big)^{\frac{p}{2}-1}$, $\forall\,(s,\omega)\in[0,\infty)\times\Omega$, it holds $P$-a.s. that

$$\frac{p\,\varepsilon}{2}\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s-s}\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})ds \le \frac{p}{2}\,\varepsilon^{\frac{p}{2}}e^{p\kappa_A}\int_0^\infty e^{-\frac{p}{2}s}ds=\varepsilon^{\frac{p}{2}}e^{p\kappa_A}.\tag{2.27}$$

Since all processes in (2.25) are RCLL ones, plugging (2.26) and (2.27) into (2.25) yields that $P$-a.s.

$$\begin{aligned}
&e^{pA_{t\wedge\tau_k}}\varphi_\varepsilon^p(t\wedge\tau_k,Y_{t\wedge\tau_k}^{m,n})+\frac{p}{4}(p-1)\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\,\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})|Z_s^{m,n}|^2ds\\
&+\frac{p}{2}(p-1)\int_{(t\wedge\tau_k,T\wedge\tau_k]}\int_{\mathcal{X}} e^{pA_s}\,\big(|Y_{s-}^{m,n}|^2\vee|Y_s^{m,n}|^2+\varepsilon e^{-s}\big)^{\frac{p}{2}-1}|U_s^{m,n}(x)|^2N_{\mathfrak{p}}(ds,dx)\\
&\quad\le g_t+\frac{p}{4}(p-1)\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})\big\|U_s^{m,n}\big\|^2_{\mathbb{L}^2_\nu}ds-p\left(M_T^{m,n,\varepsilon}-M_t^{m,n,\varepsilon}+\widetilde{M}_T^{m,n,\varepsilon}-\widetilde{M}_t^{m,n,\varepsilon}\right),\ t\in[0,T],
\end{aligned}\tag{2.28}$$

where $g_t=g_t^{m,n}\triangleq e^{p\kappa_A}\Big(\varphi_\varepsilon^p(T\wedge\tau_k,Y_{T\wedge\tau_k}^{m,n})+p\int_t^\infty\lambda(s)\theta\big(|Y_s^{m,n}|^p+\delta_n\big)ds+p\int_0^\infty\eta_s^{m,n}ds+\varepsilon^{\frac{p}{2}}\Big)$.

The Burkholder-Davis-Gundy inequality, (2.24), Lemma A.1 and Hölder's inequality imply that

$$\begin{aligned}
&E\Big[\sup_{s\in[0,T]}|M_s^{m,n,\varepsilon}|^2+\sup_{s\in[0,T]}|\widetilde{M}_s^{m,n,\varepsilon}|^2\Big]\\
&\quad\le c_0E\left[\int_0^{T\wedge\tau_k} e^{2pA_s}\left(\varphi_\varepsilon^{2p-4}(s,Y_s^{m,n})|Y_s^{m,n}|^2|Z_s^{m,n}|^2ds+\varphi_\varepsilon^{2p-4}(s,Y_{s-}^{m,n})|Y_{s-}^{m,n}|^2\big\|U_s^{m,n}\big\|^2_{\mathbb{L}^2_\nu}\right)ds\right]\\
&\quad\le c_0ke^{2p\kappa_A}E\left[\sup_{s\in[0,T\wedge\tau_k]}\varphi_\varepsilon^{2p-2}(s,Y_s^{m,n})\right]\le c_0ke^{2p\kappa_A}\left(\|Y^{m,n}\|^{2p-2}_{\mathbb{D}^p_{\mathbf{F}}}+\varepsilon^{p-1}\right)<\infty,
\end{aligned}$$

which implies that both $M_{\cdot\wedge T}^{m,n,\varepsilon}$ and $\widetilde{M}_{\cdot\wedge T}^{m,n,\varepsilon}$ are uniformly integrable martingales. Similar to (2.11), one can deduce from (2.10) that for any $t\in[0,T]$

$$E\int_{(t\wedge\tau_k,T\wedge\tau_k]}\int_{\mathcal{X}} e^{pA_s}\big(|Y_{s-}^{m,n}|^2\vee|Y_s^{m,n}|^2+\varepsilon e^{-s}\big)^{\frac{p}{2}-1}|U_s^{m,n}(x)|^2N_{\mathfrak{p}}(ds,dx)=E\int_{t\wedge\tau_k}^{T\wedge\tau_k}e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})\big\|U_s^{m,n}\big\|^2_{\mathbb{L}^2_\nu}ds.$$

Then taking expectations in (2.28) yields that

$$E\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\,\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})\left(|Z_s^{m,n}|^2+\big\|U_s^{m,n}\big\|^2_{\mathbb{L}^2_\nu}\right)ds\le\frac{4}{p(p-1)}E[g_t],\quad t\in[0,T].\tag{2.29}$$

By (2.28), it holds $P$-a.s. that

$$\begin{aligned}
\widetilde{Y}_t^{m,n,\varepsilon} &\stackrel{\triangle}{=} \sup_{s\in[t\wedge\tau_k,T\wedge\tau_k]} \big(e^{pA_s}\varphi_\varepsilon^p(s,Y_s^{m,n})\big) = \sup_{s\in[t,T]} \big(e^{pA_{s\wedge\tau_k}}\varphi_\varepsilon^p(s\wedge\tau_k,Y_{s\wedge\tau_k}^{m,n})\big) \\
&\le g_t + \frac{p}{4}(p-1)\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})\big\|U_s^{m,n}\big\|_{\mathbb{L}^2_\nu}^2 ds \\
&\quad +2p\left[\sup_{s\in[t,T]}\big|M_s^{m,n,\varepsilon}-M_t^{m,n,\varepsilon}\big| + \sup_{s\in[t,T]}\big|\widetilde{M}_s^{m,n,\varepsilon}-\widetilde{M}_t^{m,n,\varepsilon}\big|\right], \quad t\in[0,T]. \qquad (2.30)
\end{aligned}$$

Similar to (2.14), one can deduce from the Burkholder-Davis-Gundy inequality that

$$\begin{aligned}
&2p\,E\left[\sup_{s\in[t,T]}\big|M_s^{m,n,\varepsilon}-M_t^{m,n,\varepsilon}\big| + \sup_{s\in[t,T]}\big|\widetilde{M}_s^{m,n,\varepsilon}-\widetilde{M}_t^{m,n,\varepsilon}\big|\right] \\
&\quad\le c_p\,E\Bigg[\bigg(\widetilde{Y}_t^{m,n,\varepsilon}\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})|Z_s^{m,n}|^2 ds\bigg)^{\frac{1}{2}} \\
&\qquad +\bigg(\widetilde{Y}_t^{m,n,\varepsilon}\int_{(t\wedge\tau_k,T\wedge\tau_k]}\int_{\mathcal{X}} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_{s-}^{m,n})|U_s^{m,n}(x)|^2 N_{\mathfrak{p}}(ds,dx)\bigg)^{\frac{1}{2}}\Bigg] \\
&\quad\le \frac{1}{2}E\big[\widetilde{Y}_t^{m,n,\varepsilon}\big] + c_p\,E\left[\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})\Big(|Z_s^{m,n}|^2+\big\|U_s^{m,n}\big\|_{\mathbb{L}^2_\nu}^2\Big)ds\right], \quad t\in[0,T]. \qquad (2.31)
\end{aligned}$$

Since $E\big[\widetilde{Y}_t^{m,n,\varepsilon}\big] \le e^{p\kappa_A}\big\|Y^{m,n}\big\|_{\mathbb{D}^p_{\mathbf{F}}}^p + e^{p\kappa_A}\varepsilon^{\frac{p}{2}} < \infty$ by Lemma A.1, taking expectations in (2.30), we can deduce from (2.29) and (2.31) that

$$E\big[\widetilde{Y}_t^{m,n,\varepsilon}\big] \le c_p\,E[g_t], \quad t\in[0,T]. \qquad (2.32)$$

Similar to (2.18), Young's Inequality, (2.32) and (2.29) imply that

$$\begin{aligned}
&E\bigg\{\Big(\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{2A_s}|Z_s^{m,n}|^2 ds\Big)^{\frac{p}{2}}\bigg\} \le E\left\{\big(\widetilde{Y}_t^{m,n,\varepsilon}\big)^{\frac{2-p}{2}}\bigg(\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})|Z_s^{m,n}|^2 ds\bigg)^{\frac{p}{2}}\right\} \\
&\quad\le \frac{2-p}{2}E\left[\widetilde{Y}_t^{m,n,\varepsilon}\right] + \frac{p}{2}E\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{pA_s}\varphi_\varepsilon^{p-2}(s,Y_s^{m,n})|Z_s^{m,n}|^2 ds \le c_p\,E[g_t], \quad t\in[0,T]. \qquad (2.33)
\end{aligned}$$

Similarly, we can deduce that

$$E\bigg\{\Big(\int_{t\wedge\tau_k}^{T\wedge\tau_k} e^{2A_s}\big\|U_s^{m,n}\big\|_{\mathbb{L}^2_\nu}^2 ds\Big)^{\frac{p}{2}}\bigg\} \le c_p\,E[g_t], \quad t\in[0,T]. \qquad (2.34)$$

As $\big(Z_s^{m,n},U_s^{m,n}\big) \in \mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})\times\mathbb{M}^p_{\mathbf{F}}\big(\mathbb{L}^2_\nu\big)$, we see that $\int_0^\infty\Big(|Z_s^{m,n}|^2+\big\|U_s^{m,n}\big\|_{\mathbb{L}^2_\nu}^2\Big)ds < \infty$, $P$-a.s. Thus for $P$-a.s. $\omega\in\Omega$, $\tau_k(\omega)=\infty$ for some $k=k(\omega)\in\mathbb{N}$, which implies that

$$\lim_{k\to\infty}\varphi_\varepsilon(T\wedge\tau_k,Y_{T\wedge\tau_k}^{m,n}) = \varphi_\varepsilon(T,Y_T^{m,n}), \quad P\text{-a.s.},$$

although the process $Y^{m,n}$ may not be left-continuous. As $Y^{m,n}\in\mathbb{D}^p_{\mathbf{F}}$, Lemma A.1 shows that

$$\sup_{s\in[0,\infty)}\varphi_\varepsilon^p(s,Y_s^{m,n}) \le \sup_{s\in[0,\infty)}\big|Y_s^{m,n}\big|^p + \varepsilon^{\frac{p}{2}} \in \mathbb{L}^1(\mathcal{F}_\infty). \qquad (2.35)$$

Then the Dominated Convergence Theorem and (1.3) imply that

$$\lim_{k\to\infty}E\Big[\varphi_\varepsilon^p\big(T\wedge\tau_k,Y_{T\wedge\tau_k}^{m,n}\big)\Big] = E\Big[\varphi_\varepsilon^p\big(T,Y_T^{m,n}\big)\Big] \quad\text{and}\quad \lim_{T\to\infty}E\Big[\varphi_\varepsilon^p\big(T,Y_T^{m,n}\big)\Big] = E\Big[\big|\xi_m-\xi_n\big|^p\Big].$$

Adding up (2.32)-(2.34), letting $k\to\infty$ and then letting $T\to\infty$, we can deduce from the Monotone Convergence Theorem that

$$E\left[\Xi^{m,n}_t\right] \le c_p\, e^{p\kappa_A}\left(E\Big[\big|\xi_m-\xi_n\big|^p\Big]+E\int_t^\infty \lambda(s)\theta\big(\Xi^{m,n}_s+\delta_n\big)ds+E\left[\int_0^\infty \eta^{m,n}_s ds\right]+\varepsilon^{\frac{p}{2}}\right),\quad t\in[0,\infty),$$

where $\Xi^{m,n}_t \stackrel{\triangle}{=} \sup\limits_{s\in[t,\infty)}|Y^{m,n}_s|^p+\Big(\int_t^\infty |Z^{m,n}_s|^2 ds\Big)^{\frac{p}{2}}+\Big(\int_t^\infty \big\|U^{m,n}_s\big\|^2_{\mathbb{L}^2_\nu} ds\Big)^{\frac{p}{2}}$. Then letting $\varepsilon\to 0$, one can deduce from Fubini Theorem, the concavity of $\theta$ and Jensen's Inequality that

$$\begin{aligned}E\left[\Xi^{m,n}_t\right] &\le c_p\, e^{p\kappa_A}\left(E\Big[\big|\xi_m-\xi_n\big|^p\Big]+\int_t^\infty \lambda(s)E\left[\theta\big(\Xi^{m,n}_s+\delta_n\big)\right]ds+E\left[\int_0^\infty \eta^{m,n}_s ds\right]\right)\\ &\le c_p\, e^{p\kappa_A}\left(E\Big[\big|\xi_m-\xi_n\big|^p\Big]+\int_t^\infty \lambda(s)\theta\big(E\left[\Xi^{m,n}_s\right]+E[\delta_n]\big)ds+E\left[\int_0^\infty \eta^{m,n}_s ds\right]\right),\quad t\in[0,\infty).\end{aligned}$$

Hence, it holds for any $n\in\mathbb{N}$ and $t\in[0,\infty)$ that

$$\sup_{m>n}E\left[\Xi^{m,n}_t\right] \le c_p\, e^{p\kappa_A}\left(\sup_{m>n}E\Big[\big|\xi_m-\xi_n\big|^p\Big]+\int_t^\infty \lambda(s)\theta\Big(\sup_{m>n}E\left[\Xi^{m,n}_s\right]+E[\delta_n]\Big)ds+\sup_{m>n}E\left[\int_0^\infty \eta^{m,n}_s ds\right]\right). \quad (2.36)$$

Since $\{\xi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{L}^p(\mathcal{F}_\infty)$, one has

$$\lim_{n\to\infty}\sup_{m>n}E\Big[\big|\xi_m-\xi_n\big|^p\Big]=0. \quad (2.37)$$

If $\lambda$ is trivial, (i.e. $\int_0^\infty \lambda(t)dt=0$), then $\int_0^\infty \lambda(s)\theta\Big(\sup\limits_{m>n}E\left[\Xi^{m,n}_s\right]+E[\delta_n]\Big)ds=0$. Taking $t=0$ and letting $n\to\infty$ in (2.36), we see from (2.37) and (2.22) that

$$\lim_{n\to\infty}\sup_{m>n}E\left[\Xi^{m,n}_0\right]=0. \quad (2.38)$$

On the other hand, we assume that $\lambda$ is non-trivial. Since $\lambda\in\mathbb{L}^1_+[0,\infty)$ and since

$$\sup_{m>n}E\left[\Xi^{m,n}_s\right]+E[\delta_n]\le\left\{2\sup_{i\in\mathbb{N}}\Big(\|Y^i\|_{\mathbb{D}^p_{\mathbf{F}}}+\|Z^i\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})}+\|U^i\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{L}^2_\nu)}\Big)\right\}^p+\sup_{i\in\mathbb{N}}E\big[\delta_i\big]<\infty,\ \ \forall\,(s,n)\in[0,\infty)\times\mathbb{N} \quad (2.39)$$

by (2.23), Fatou's Lemma, the monotonicity and the continuity of $\theta$[5] imply that

$$\begin{aligned}\varlimsup_{n\to\infty}\int_t^\infty \lambda(s)\theta\big(\sup_{m>n}E\left[\Xi^{m,n}_s\right]+E[\delta_n]\big)ds &\le \int_t^\infty \lambda(s)\varlimsup_{n\to\infty}\theta\big(\sup_{m>n}E\left[\Xi^{m,n}_s\right]+E[\delta_n]\big)ds\\ &\le \int_t^\infty \lambda(s)\theta\left(\varlimsup_{n\to\infty}\sup_{m>n}E\left[\Xi^{m,n}_s\right]\right)ds,\quad t\in[0,\infty). \end{aligned}\quad (2.40)$$

Letting $n\to\infty$ in (2.36), we can deduce from (2.37), (2.22) and (2.40) that

$$\varlimsup_{n\to\infty}\sup_{m>n}E\left[\Xi^{m,n}_t\right]\le c_p\, e^{p\kappa_A}\int_t^\infty \lambda(s)\theta\left(\varlimsup_{n\to\infty}\sup_{m>n}E\left[\Xi^{m,n}_s\right]\right)ds,\quad t\in[0,\infty).$$

As $\theta:[0,\infty)\mapsto[0,\infty)$ is an increasing concave function, it is easy to see that either $\theta\equiv 0$ or $\theta(t)>0$ for any $t>0$. Moreover, one can deduce from (2.39) that the function $\mu(t)\stackrel{\triangle}{=}\varlimsup\limits_{n\to\infty}\sup\limits_{m>n}E\left[\Xi^{m,n}_t\right]$, $t\in[0,\infty)$ is bounded. Then Lemma A.3 and (2.39) imply that $\lim\limits_{n\to\infty}\sup\limits_{m>n}E\left[\Xi^{m,n}_t\right]=0$, $\forall\,t\in[0,\infty)$. Therefore, (2.38) always holds, which shows that $\big\{(Y^n,Z^n,U^n)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{S}^p_{\mathbf{F}}$. □

**Proposition 2.3.** *Let $(\xi,f)$ be a parameter pair such that $\xi\in\mathbb{L}^\infty(\mathcal{F}_\infty)$. If the generator $f$ satisfies (H1) for each $(t,\omega)\in[0,\infty)\times\Omega$ and satisfies (H2)-(H4) for $dt\otimes dP$-a.e. $(t,\omega)\in[0,\infty)\times\Omega$, then the BSDEJ$(\xi,f)$ has a solution $(Y,Z,U)\in\mathbb{D}^\infty_{\mathbf{F}}\times\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})\times\mathbb{M}^p_{\mathbf{F}}\big(\mathbb{L}^2_\nu\big)$.*

[5] It is known that any $\mathbb{R}$-valued concave function is also a continuous function.

**Proof:** We make the following settings first:

• let $\psi:\mathbb{R}^l\mapsto[0,1]$ be a smooth function that equals to 1 (resp. 0) when $|x|\le R-1$ $\big($resp. $|x|\ge R\big)$, where

$$R\stackrel{\triangle}{=}2+\exp\bigg\{2\bigg\|\int_0^\infty\beta_s ds\bigg\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}+\int_0^\infty\Big(\frac{c(s)}{2}+2c^2(s)\Big)ds\bigg\}\cdot\sqrt{\|\xi\|^2_{\mathbb{L}^\infty(\mathcal{F}_\infty)}+2\bigg\|\int_0^\infty\beta_s ds\bigg\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}+\int_0^\infty c(s)ds}.$$

• Let $\rho:\mathbb{R}^{l+l\times d}\mapsto\mathbb{R}^+$ be a smooth function that vanishes outside the unit open ball $\mathcal{B}_1(0)$ of $\mathbb{R}^{l+l\times d}$ and satisfies $\int_{\mathbb{R}^{l+l\times d}}\rho(x)dx=1$. For any $r\in(0,\infty)$, we set $\rho_r(x)\stackrel{\triangle}{=}r^{l(1+d)}\rho(rx),\ \forall\,x\in\mathbb{R}^{l+l\times d}$.

• Let $\big\{O^k_i\big\}_{i=1}^{2^k}$, $k\in\mathbb{N}$ be partitions[6] of $\overline{\mathcal{B}}_1(0)$ such that $\overline{O}^k_i=\overline{O}^{k+1}_{2i-1}\cup\overline{O}^{k+1}_{2i}$ holds for each $O^k_i$. For any $k\in\mathbb{N}$ and $i=1,\cdots,2^k$, we pick up a $(y^k_i,z^k_i)\in O^k_i$ with $y^k_i\in\mathbb{R}^l$, and let $\big\|O^k_i\big\|$ denote the volume of $O^k_i$.

**(1)** Fix $n\in\mathbb{N}$. Clearly, $\beta^n_t\stackrel{\triangle}{=}\dfrac{ne^{-t}}{(ne^{-t})\vee\beta_t}$, $t\in[0,\infty)$ is an $\mathbf{F}$-progressively measurable process, which implies that the function

$$f^0_n(t,\omega,y,z,u)\stackrel{\triangle}{=}\beta^n_t(\omega)\psi(y)f\big(t,\omega,y,\pi_n(z),\pi_n(u)\big),\quad\forall\,(t,\omega,y,z,u)\in[0,\infty)\times\Omega\times\mathbb{R}^l\times\mathbb{R}^{l\times d}\times\mathbb{L}^2_\nu$$

is $\mathscr{P}\times\mathscr{B}\big(\mathbb{R}^l\big)\times\mathscr{B}\big(\mathbb{R}^{l\times d}\big)\times\mathscr{B}\big(\mathbb{L}^2_\nu\big)/\mathscr{B}(\mathbb{R}^l)$-measurable. Then we fix $(t,\omega,y,z,u)\in[0,\infty)\times\Omega\times\mathbb{R}^l\times\mathbb{R}^{l\times d}\times\mathbb{L}^2_\nu$ and define

$$f_n(t,\omega,y,z,u)\stackrel{\triangle}{=}\big(f^0_n(t,\omega,\cdot,\cdot,u)*\rho_n\big)(y,z).$$

By (H1), the continuity of mapping $f(t,\omega,\cdot,\cdot,u)$ implies that of mapping $f^0_n(t,\omega,\cdot,\cdot,u)$. Hence, $f_n(t,\omega,y,z,u)$ is indeed a Riemann integral:

$$\begin{aligned}f_n(t,\omega,y,z,u)&=\int_{|(\tilde{y},\tilde{z})|\le 1}f^0_n\Big(t,\omega,y-\frac{1}{n}\tilde{y},z-\frac{1}{n}\tilde{z},u\Big)\rho(\tilde{y},\tilde{z})d\tilde{y}d\tilde{z}\\&=\lim_{k\to\infty}\sum_{m=1}^{2^i}f^0_n\Big(t,\omega,y-\frac{1}{n}y^k_i,z-\frac{1}{n}z^k_i,u\Big)\rho(y^k_i,z^k_i)\big\|O^k_i\big\|,\end{aligned}\tag{2.41}$$

from which one can deduce that $f_n$ is also $\mathscr{P}\times\mathscr{B}\big(\mathbb{R}^l\big)\times\mathscr{B}\big(\mathbb{R}^{l\times d}\big)\times\mathscr{B}\big(\mathbb{L}^2_\nu\big)/\mathscr{B}(\mathbb{R}^l)$-measurable.

Let $c_n(t)\stackrel{\triangle}{=}(1+R)ne^{-t}+2nc(t)$, $t\in[0,\infty)$. Clearly, $c_n(\cdot)\in\mathbb{L}^1_+[0,\infty)\cap\mathbb{L}^2_+[0,\infty)$. It follows from (H2) and (H3) that $dt\otimes dP$-a.e.

$$\begin{aligned}|f^0_n(t,y,z,u)|&\le\beta^n_t\psi(y)\Big|f\big(t,y,\pi_n(z),\pi_n(u)\big)\Big|\le\Big|f\big(t,y,\pi_n(z),\pi_n(u)\big)-f\big(t,y,\pi_n(z),0\big)\Big|+\beta^n_t\psi(y)\big|f\big(t,y,\pi_n(z),0\big)\big|\\&\le c(t)\big\|\pi_n(u)\big\|_{\mathbb{L}^2_\nu}+\beta^n_t\psi(y)\big(1+|y|\big)\beta_t+c(t)\big|\pi_n(z)\big|\le c_n(t),\quad\forall\,(y,z,u)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}\times\mathbb{L}^2_\nu,\end{aligned}$$

which implies that $dt\otimes dP$-a.e.

$$\begin{aligned}|f_n(t,y_1,z_1,u)-f_n(t,y_2,z_2,u)|&=\bigg|\int_{\mathbb{R}^{l+l\times d}}\Big(\int_0^1\Big\langle\big(y_1-y_2,z_1-z_2\big),\nabla\rho_n\big(y_\alpha-\tilde{y},z_\alpha-\tilde{z}\big)\Big\rangle d\alpha\Big)f^0_n(t,\tilde{y},\tilde{z},u)d\tilde{y}d\tilde{z}\bigg|\\&\le c_n(t)\int_0^1\int_{\mathbb{R}^{l+l\times d}}\big|(y_1-y_2,z_1-z_2)\big|\cdot\big|\nabla\rho_n\big(y_\alpha-\tilde{y},z_\alpha-\tilde{z}\big)\big|d\tilde{y}d\tilde{z}d\alpha\\&\le\kappa^n_\rho\,c_n(t)\big(|y_1-y_2|+|z_1-z_2|\big),\ \ \forall\,(y_1,z_1),(y_2,z_2)\in\mathbb{R}^l\times\mathbb{R}^{l\times d},\ \ \forall\,u\in\mathbb{L}^2_\nu,\end{aligned}\tag{2.42}$$

where $y_\alpha\stackrel{\triangle}{=}\alpha y_1+(1-\alpha)y_2$, $z_\alpha\stackrel{\triangle}{=}\alpha z_1+(1-\alpha)z_2$, $\forall\,\alpha\in(0,1)$, and $\kappa^n_\rho\stackrel{\triangle}{=}\int_{\mathbb{R}^{l+l\times d}}|\nabla\rho_n(x)|dx<\infty$ is a constant determined by $\rho$ and $n$.

[6] We say that $\{O_i\}_{i=1}^m$ is a partition of the unit closed ball $\overline{\mathcal{B}}_1(0)$ of $\mathbb{R}^{l+l\times d}$ if $O_i$, $i=1,\cdots,m$ are single-connected, open subsets of $\mathcal{B}_1(0)$ that are pairwisely disjoint, and if $\cup_{i=1}^m\overline{O}_i=\overline{\mathcal{B}}_1(0)$.

Moreover, (2.41), (H3) and Lemma A.5 imply that $dt \otimes dP$-a.e.

$$
\begin{aligned}
&|f_n(t,y,z,u_1) - f_n(t,y,z,u_2)| \\
&\quad = \bigg| \int_{|(\tilde{y},\tilde{z})|\le 1} \beta^n_t \psi\Big(y - \frac{1}{n}\tilde{y}\Big) \Big( f\Big(t, y - \frac{1}{n}\tilde{y}, \pi_n\Big(z - \frac{1}{n}\tilde{z}\Big), \pi_n(u_1)\Big) - f\Big(t, y - \frac{1}{n}\tilde{y}, \pi_n\Big(z - \frac{1}{n}\tilde{z}\Big), \pi_n(u_2)\Big)\Big) \rho(\tilde{y},\tilde{z}) d\tilde{y} d\tilde{z} \bigg| \\
&\quad \le \int_{|(\tilde{y},\tilde{z})|\le 1} \Big| f\Big(t, y - \frac{1}{n}\tilde{y}, \pi_n\Big(z - \frac{1}{n}\tilde{z}\Big), \pi_n(u_1)\Big) - f\Big(t, y - \frac{1}{n}\tilde{y}, \pi_n\Big(z - \frac{1}{n}\tilde{z}\Big), \pi_n(u_2)\Big) \Big| \rho(\tilde{y},\tilde{z}) d\tilde{y} d\tilde{z} \\
&\quad \le c(t) \big\| \pi_n(u_1) - \pi_n(u_2) \big\|_{\mathbb{L}^2_\nu} \le c(t) \big\| u_1 - u_2 \big\|_{\mathbb{L}^2_\nu}, \quad \forall\, (y,z) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}, \quad \forall\, u_1, u_2 \in \mathbb{L}^2_\nu,
\end{aligned}
$$

which together with (2.42) shows that $f_n$ satisfies (1.4) with $\phi_1(t) = \kappa^n_\rho\, c_n(t)$ and $\phi_2(t) = \phi_1(t) + c(t)$, $t \in [0,\infty)$.

Moreover, by (2.41) and (H2), it holds $dt \otimes dP$-a.e. that

$$
\begin{aligned}
\big| f_n(t,0,0,0) \big| &\le \int_{|(\tilde{y},\tilde{z})|\le 1} \beta^n_t \Big| f\Big(t, -\frac{1}{n}\tilde{y}, \pi_n\Big(-\frac{1}{n}\tilde{z}\Big), 0\Big) \Big| \rho(\tilde{y},\tilde{z}) d\tilde{y} d\tilde{z} \\
&\le \int_{|(\tilde{y},\tilde{z})|\le 1} \Big( \beta^n_t \Big(1 + \frac{1}{n}|\tilde{y}|\Big) \beta_t + c(t) \Big| \pi_n\Big(-\frac{1}{n}\tilde{z}\Big) \Big| \Big) \rho(\tilde{y},\tilde{z}) d\tilde{y} d\tilde{z} \\
&\le \int_{|(\tilde{y},\tilde{z})|\le 1} \Big( (n+1)e^{-t} + \frac{1}{n} c(t) \Big) \rho(\tilde{y},\tilde{z}) d\tilde{y} d\tilde{z} \le (n+1)e^{-t} + \frac{1}{n} c(t).
\end{aligned}
$$

which implies that $E\Big[\big(\int_0^\infty |f(t,0,0,0)|dt\big)^2\Big] \le \big(n + 1 + \frac{1}{n}\int_0^\infty c(t)dt\big)^2 < \infty$. Therefore, we know from Lemma 1.1 that the BSDEJ$(\xi, f_n)$ has a unique solution $(Y^n, Z^n, U^n) \in \mathbb{S}^2_{\mathbf{F}}$.

**(2)** Now, we define $a_t \stackrel{\triangle}{=} 4\beta_t + c(t) + 4c^2(t)$ and $A_t = \int_0^t a_s ds$, $t \in [0,\infty)$. It easily follows from (h1) and (h2) that $A_\infty \in \mathbb{L}^\infty_+(\mathcal{F}_\infty)$ with $\kappa_A \stackrel{\triangle}{=} \big\| A_\infty \big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)} \le 4 \big\| \int_0^\infty \beta_s ds \big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)} + \int_0^\infty \Big(c(s) + 4c^2(s)\Big) ds$.

Fix $n \in \mathbb{N}$ and $0 \le t < T < \infty$. Applying Itô's formula to $e^{A_s}|Y^n_s|^2$ over the interval $[t,T]$ yields that

$$
\begin{aligned}
&e^{A_t}|Y^n_t|^2 + \int_t^T e^{A_s}|Z^n_s|^2 ds + \int_{(t,T]} \int_{\mathcal{X}} e^{A_s}|U^n_s(x)|^2 N_{\mathfrak{p}}(ds,dx) \\
&\quad \le e^{A_T}|Y^n_T|^2 + 2\int_t^T e^{A_s} \big\langle Y^n_s, f_n(s, Y^n_s, Z^n_s, U^n_s) \big\rangle ds - \int_t^T a_s e^{A_s} |Y^n_s|^2 ds - 2\big(M_T - M_t + \widetilde{M}_T - \widetilde{M}_t\big), \; P\text{-a.s.}, \quad (2.43)
\end{aligned}
$$

where $M_r \stackrel{\triangle}{=} \int_0^r e^{A_s} \langle Y^n_s, Z^n_s\, dB_s \rangle$ and $\widetilde{M}_r \stackrel{\triangle}{=} \int_{(0,r]} \int_{\mathcal{X}} e^{A_s} \langle Y^n_{s-}, U^n_s(x) \rangle \widetilde{N}_{\mathfrak{p}}(ds,dx)$, $\forall\, r \in [0,\infty)$. One can deduce from (H2) and (H3) that $dt \otimes dP$-a.e.

$$
\begin{aligned}
2\langle Y^n_s, f_n(s, Y^n_s, Z^n_s, U^n_s) \rangle &= 2\int_{|(y,z)|<1} \beta^n_s \psi\Big(Y^n_s - \frac{1}{n}y\Big) \Big\langle Y^n_s, f\Big(s, Y^n_s - \frac{1}{n}y, \pi_n\Big(Z^n_s - \frac{1}{n}z\Big), \pi_n(U^n_s)\Big) \Big\rangle \rho(y,z) dy dz \\
&\le 2\int_{|(y,z)|<1} |Y^n_s| \left[ \Big(1 + \Big|Y^n_s - \frac{1}{n}y\Big|\Big) \beta_s + c(s) \Big| \pi_n\Big(Z^n_s - \frac{1}{n}z\Big) \Big| + c(s) \big\| \pi_n(U^n_s) \big\|_{\mathbb{L}^2_\nu} \right] \rho(y,z) dy dz \\
&\le 2|Y^n_s| \Big( (2 + |Y^n_s|)\beta_s + c(s)\Big[1 + |Z^n_s| + \|U^n_s\|_{\mathbb{L}^2_\nu}\Big] \Big) \\
&\le 2\beta_s + c(s) + a_s|Y^n_s|^2 + \frac{1}{2}|Z^n_s|^2 + \frac{1}{2}\big\|U^n_s\big\|^2_{\mathbb{L}^2_\nu}. \qquad (2.44)
\end{aligned}
$$

Moreover, Burkholder-Davis-Gundy inequality and Hölder's inequality imply that

$$
\begin{aligned}
E\Big[\sup_{s\in[0,\infty)} |M_s| + \sup_{s\in[0,\infty)} |\widetilde{M}_s|\Big] &\le c_0 E\left[ \Big( \int_0^\infty e^{2A_s} |Y^n_s|^2 |Z^n_s|^2 ds \Big)^{\frac{1}{2}} + \Big( \int_0^\infty e^{2A_s} |Y^n_{s-}|^2 \big\|U^n_s\big\|^2_{\mathbb{L}^2_\nu} ds \Big)^{\frac{1}{2}} \right] \\
&\le c_0 e^{\kappa_A} E\left[ \sup_{s\in[0,\infty)} |Y^n_s| \Big( \int_0^\infty |Z^n_s|^2 ds \Big)^{\frac{1}{2}} + \sup_{s\in[0,\infty)} |Y^n_s| \Big( \int_0^\infty \big\|U^n_s\big\|^2_{\mathbb{L}^2_\nu} ds \Big)^{\frac{1}{2}} \right] \\
&\le c_0 e^{\kappa_A} \|Y^n\|_{\mathbb{D}^2_{\mathbf{F}}} \Big( \|Z^n\|_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{R}^{l\times d})} + \|U^n\|_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{L}^2_\nu)} \Big) < \infty,
\end{aligned}
$$

which shows that both $M$ and $\widetilde{M}$ are uniformly integrable martingales. Hence, taking conditional expectation $E[\cdot|\mathcal{F}_t]$ in (2.43), we can deduce from (2.44) that $P$-a.s.

$$e^{A_t}|Y^n_t|^2+\frac{1}{2}E\left[\int_t^T e^{A_s}\left(|Z^n_s|^2+\big\|U^n_s\big\|^2_{\mathbb{L}^2_\nu}\right)ds\Big|\mathcal{F}_t\right]\le e^{\kappa_A}\left(E\big[|Y^n_T|^2\big|\mathcal{F}_t\big]+2\left\|\int_0^\infty\beta_s ds\right\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}+\int_0^\infty c(s)ds\right),\ (2.45)$$

where we used the fact that

$$E\left[\int_{(t,T]}\int_{\mathcal{X}} e^{A_s}|U^n_s(x)|^2 N_{\mathfrak{p}}(ds,dx)\Big|\mathcal{F}_t\right]=E\left[\int_t^T\int_{\mathcal{X}} e^{A_s}|U^n_s(x)|^2\nu(dx)\,ds\Big|\mathcal{F}_t\right]=E\left[\int_t^T e^{A_s}\big\|U^n_s\big\|^2_{\mathbb{L}^2_\nu}ds\Big|\mathcal{F}_t\right],\ P\text{-a.s.}$$

Since $Y^n\in\mathbb{D}^2_{\mathbf{F}}$, the Dominated Convergence Theorem implies that

$$\lim_{T\to\infty}E\big[|Y^n_T|^2\big|\mathcal{F}_t\big]=E\big[|Y^n_\infty|^2\big|\mathcal{F}_t\big]=E\big[|\xi|^2\big|\mathcal{F}_t\big]\le\big\|\xi\big\|^2_{\mathbb{L}^\infty(\mathcal{F}_\infty)},\quad P\text{-a.s.}$$

Hence, as $T\to\infty$ in (2.45), the Monotone Convergence Theorem gives that

$$\begin{aligned}&e^{A_t}|Y^n_t|^2+\frac{1}{2}E\left[\int_t^\infty e^{A_s}\left(|Z^n_s|^2+\big\|U^n_s\big\|^2_{\mathbb{L}^2_\nu}\right)ds\Big|\mathcal{F}_t\right]\\ &\quad\le\ e^{\kappa_A}\left(\big\|\xi\big\|^2_{\mathbb{L}^\infty(\mathcal{F}_\infty)}+2\left\|\int_0^\infty\beta_s ds\right\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}+\int_0^\infty c(s)ds\right)\le(R-2)^2,\quad P\text{-a.s.},\end{aligned}$$

which together with the right-continuity of $Y^n$ implies that

$$\|Y^n\|_{\mathbb{D}^\infty_{\mathbf{F}}}\le R-2\quad\text{and}\quad\big\|Z^n\big\|^2_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{R}^{l\times d})}+\big\|U^n\big\|^2_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{L}^2_\nu)}\le 2(R-2)^2,\quad\forall\,n\in\mathbb{N}.\tag{2.46}$$

**(3)** Except on a $dt\otimes dP$-null set $\widetilde{\mathcal{N}}_1$ of $[0,\infty)\times\Omega$, We may assume that (H2)-(H4) hold and that $|Y^n_t|\le R-2$, $\forall\,n\in\mathbb{N}$. Fix $(t,\omega)\in\widetilde{\mathcal{N}}^c_1$ and fix $m,\,n\in\mathbb{N}$ with $m>n$. By (2.41),

$$\begin{aligned}&|Y^{m,n}_t|^{p-1}\big\langle\mathscr{D}(Y^{m,n}_t),f_m(s,Y^m_t,Z^m_t,U^m_t)-f_n(s,Y^n_t,Z^n_t,U^n_t)\big\rangle\\ &\quad=\ \int_{|(\tilde y,\tilde z)|<1}|Y^{m,n}_t|^{p-1}\big\langle\mathscr{D}(Y^{m,n}_t),\beta^m_t h^m_t(\tilde y,\tilde z)-\beta^n_t h^n_t(\tilde y,\tilde z)\big\rangle\rho(\tilde y,\tilde z)d\tilde y d\tilde z,\end{aligned}\tag{2.47}$$

where $h^n_t(\tilde y,\tilde z)\stackrel{\triangle}{=}f\big(s,Y^n_t-\frac{1}{n}\tilde y,\pi_n(Z^n_t-\frac{1}{n}\tilde z),\pi_n(U^n_t)\big)$. Next, fix $(\tilde y,\tilde z)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}$ with $|(\tilde y,\tilde z)|<1$. We set $(\widetilde{y}_{m,n},\widetilde{z}_{m,n})\stackrel{\triangle}{=}\Big(\big(\frac{1}{m}-\frac{1}{n}\big)\tilde y,\big(\frac{1}{m}-\frac{1}{n}\big)\tilde z\Big)$ and consider the following decomposition:

$$\begin{aligned}&|Y^{m,n}_t|^{p-1}\big\langle\mathscr{D}(Y^{m,n}_t),\beta^m_t h^m_t(\tilde y,\tilde z)-\beta^n_t h^n_t(\tilde y,\tilde z)\big\rangle\\ &\quad=\ \beta^m_t\big|Y^{m,n}_t-\widetilde{y}_{m,n}\big|^{p-1}\big\langle\mathscr{D}\left(Y^{m,n}_t-\widetilde{y}_{m,n}\right),h^m_t(\tilde y,\tilde z)-h^n_t(\tilde y,\tilde z)\big\rangle\\ &\qquad+\beta^m_t\big\langle|Y^{m,n}_t|^{p-1}\mathscr{D}\left(Y^{m,n}_t\right)-\big|Y^{m,n}_t-\widetilde{y}_{m,n}\big|^{p-1}\mathscr{D}\left(Y^{m,n}_t-\widetilde{y}_{m,n}\right),h^m_t(\tilde y,\tilde z)-h^n_t(\tilde y,\tilde z)\big\rangle\\ &\qquad+|Y^{m,n}_t|^{p-1}\big\langle\mathscr{D}\left(Y^{m,n}_t\right),(\beta^m_t-\beta^n_t)h^n_t(\tilde y,\tilde z)\big\rangle\stackrel{\triangle}{=}I^1_t(\tilde y,\tilde z)+I^2_t(\tilde y,\tilde z)+I^3_t(\tilde y,\tilde z).\end{aligned}$$

It follows from (H4) that

$$\begin{aligned}I^1_t(\tilde y,\tilde z)\ &\le\ \lambda(t)\theta(|Y^{m,n}_t-\widetilde{y}_{m,n}|^p)+\Lambda_t|Y^{m,n}_t-\widetilde{y}_{m,n}|^p+\widetilde{\Lambda}_t|Y^{m,n}_t-\widetilde{y}_{m,n}|^{p-1}\Big(\big|\pi_m(Z^m_t-\tfrac{1}{m}\tilde z)-\pi_n(Z^n_t-\tfrac{1}{n}\tilde z)\big|\\ &\qquad+\big\|\pi_m(U^m_t)-\pi_n(U^n_t)\big\|_{\mathbb{L}^2_\nu}\Big).\end{aligned}\tag{2.48}$$

Applying Lemma A.2 with $a=|Y^{m,n}_t-\widetilde{y}_{m,n}|$ and $b=|Y^{m,n}_t|$ yields that

$$|Y^{m,n}_t-\widetilde{y}_{m,n}|^{p-1}\ \le\ \big|Y^{m,n}_t\big|^{p-1}+|\widetilde{y}_{m,n}|^{p-1}\le\big|Y^{m,n}_t\big|^{p-1}+n^{1-p}\tag{2.49}$$

$$\text{and}\qquad|Y^{m,n}_t-\widetilde{y}_{m,n}|^p\ \le\ \big|Y^{m,n}_t\big|^p+p\Big(|Y^{m,n}_t|+|\widetilde{y}_{m,n}|\Big)^{p-1}\big|\widetilde{y}_{m,n}\big|\le\big|Y^{m,n}_t\big|^p+\delta_n\tag{2.50}$$

with $\delta_n \triangleq \frac{p}{n}(2R-3)^{p-1}$. Moreover, Lemma A.5 shows that

$$\begin{aligned}
&\big|\pi_m(Z^m_t-\tfrac{1}{m}\tilde z)-\pi_n(Z^n_t-\tfrac{1}{n}\tilde z)\big| \le \big|\pi_m(Z^m_t-\tfrac{1}{m}\tilde z)-\pi_m(Z^n_t-\tfrac{1}{n}\tilde z)\big|+\big|\pi_m(Z^n_t-\tfrac{1}{n}\tilde z)-\pi_n(Z^n_t-\tfrac{1}{n}\tilde z)\big| \\
&\qquad \le |Z^{m,n}_t-\widetilde z_{m,n}|+\mathbf{1}_{\{|Z^n_t-\frac{1}{n}\tilde z|>n\}}|Z^n_t-\tfrac{1}{n}\tilde z| \le |Z^{m,n}_t|+\tfrac{2}{n}+\mathbf{1}_{\{|Z^n_t|>n-1\}}|Z^n_t|. \qquad (2.51)
\end{aligned}$$

Similarly, we have

$$\big\|\pi_m(U^m_t)-\pi_n(U^n_t)\big\|_{\mathbb{L}^2_\nu} \le \|U^{m,n}_t\|_{\mathbb{L}^2_\nu}+\mathbf{1}_{\{\|U^n_t\|_{\mathbb{L}^2_\nu}>n\}}\|U^n_t\|_{\mathbb{L}^2_\nu}. \qquad (2.52)$$

Putting (2.49)-(2.52) back into (2.48), we can deduce from the monotonicity of function $\theta$ that

$$\begin{aligned}
I^1_t(\tilde y,\tilde z) \;&\le\; \lambda(t)\theta\big(|Y^{m,n}_t|^p+\delta_n\big)+\Lambda_t\big(|Y^{m,n}_t|^p+\delta_n\big)+\widetilde\Lambda_t\left(|Y^{m,n}_t|^{p-1}+n^{1-p}\right)\Big(\Psi^n_t+|Z^{m,n}_t|+\|U^{m,n}_t\|_{\mathbb{L}^2_\nu}\Big) \\
&\le\; \lambda(t)\theta\big(|Y^{m,n}_t|^p+\delta_n\big)+\Lambda_t|Y^{m,n}_t|^p+\delta_n\Lambda_t+\big[1+(2R-4)^{p-1}\big]\widetilde\Lambda_t\Psi^n_t \\
&\quad +\widetilde\Lambda_t|Y^{m,n}_t|^{p-1}\Big(|Z^{m,n}_t|+\|U^{m,n}_t\|_{\mathbb{L}^2_\nu}\Big)+n^{1-p}\Big(\frac{1}{2}\widetilde\Lambda^2_t+|Z^{m,n}_t|^2+\big\|U^{m,n}_t\big\|^2_{\mathbb{L}^2_\nu}\Big), \qquad (2.53)
\end{aligned}$$

where $\Psi^n_t \triangleq \frac{2}{n}+\mathbf{1}_{\{|Z^n_t|>n-1\}}|Z^n_t|+\mathbf{1}_{\{\|U^n_t\|_{\mathbb{L}^2_\nu}>n\}}\|U^n_t\|_{\mathbb{L}^2_\nu}$.

On the other hand, one can deduce from (H2) and (H3) that

$$\begin{aligned}
|h^n_t(\tilde y,\tilde z)| \;&\le\; \big(1+|Y^n_t-\tfrac{1}{n}\tilde y|\big)\beta_t+c(t)\big|\pi_n(Z^n_t-\tfrac{1}{n}\tilde z)\big|+c(t)\big\|\pi_n(U^n_t)\big\|_{\mathbb{L}^2_\nu} \\
&\le\; R\beta_t+c(t)\big(1+|Z^n_t|+\|U^n_t\|_{\mathbb{L}^2_\nu}\big), \qquad (2.54)
\end{aligned}$$

which together with Lemma A.6 yield that

$$\begin{aligned}
I^2_t(\tilde y,\tilde z) \;&\le\; \Big||Y^{m,n}_t|^{p-1}\mathscr{D}\big(Y^{m,n}_t\big)-\big|Y^{m,n}_t-\widetilde y_{m,n}\big|^{p-1}\mathscr{D}\big(Y^{m,n}_t-\widetilde y_{m,n}\big)\Big|\Big(\big|h^m_t(\tilde y,\tilde z)\big|+\big|h^n_t(\tilde y,\tilde z)\big|\Big) \\
&\le\; (1+2^{p-1})n^{1-p}\Big[2R\beta_t+c(t)\big(2+|Z^m_t|+|Z^n_t|+\|U^m_t\|_{\mathbb{L}^2_\nu}+\|U^n_t\|_{\mathbb{L}^2_\nu}\big)\Big] \triangleq \widetilde I^2_t. \qquad (2.55)
\end{aligned}$$

Since $0<\beta^n_t \le \beta^m_t \le 1,\ \forall\, t\in[0,\infty)$, (2.54) also implies that

$$I^3_t(\tilde y,\tilde z) \le (2R-4)^{p-1}(1-\beta^n_t)\Big[R\beta_t+c(t)\big(1+|Z^n_t|+\|U^n_t\|_{\mathbb{L}^2_\nu}\big)\Big] \triangleq \widetilde I^3_t. \qquad (2.56)$$

Plugging (2.53), (2.55) and (2.56) back into (2.47) shows that (2.21) is satisfied with $\delta_n=\frac{p}{n}(2R-3)^{p-1}$ and

$$\eta^{m,n}_t=\delta_n\Lambda_t+\big[1+(2R-4)^{p-1}\big]\widetilde\Lambda_t\Psi^n_t+n^{1-p}\Big(\frac{1}{2}\widetilde\Lambda^2_t+|Z^{m,n}_t|^2+\big\|U^{m,n}_t\big\|^2_{\mathbb{L}^2_\nu}\Big)+\widetilde I^2_t+\widetilde I^3_t, \quad t\in[0,\infty).$$

Hölder's Inequality and (2.46) give rise to the following four estimates:

*1.*
$$\begin{aligned}
\sup_{m>n} E\left[\int_0^\infty \eta^{m,n}_t dt\right] \;&\le\; \delta_n\left\|\int_0^\infty \Lambda_t dt\right\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}+\big[1+(2R-4)^{p-1}\big]E\int_0^\infty \widetilde\Lambda_t\Psi^n_t dt \\
&\quad +n^{1-p}\left(\frac{1}{2}\left\|\int_0^\infty \widetilde\Lambda^2_t dt\right\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}+8(R-2)^2\right)+E\int_0^\infty \big(\widetilde I^2_t+\widetilde I^3_t\big)dt. \qquad (2.57)
\end{aligned}$$

*2.*
$$\begin{aligned}
E\int_0^\infty \widetilde\Lambda_t\Psi^n_t dt &\le \frac{2}{n}C^1_{\widetilde\Lambda}+C^{2+\varpi}_{\widetilde\Lambda}\left[\left(E\int_0^\infty \mathbf{1}_{\{|Z^n_t|>n-1\}}|Z^n_t|^{\frac{2+\varpi}{1+\varpi}}dt\right)^{\frac{1+\varpi}{2+\varpi}}+\left(E\int_0^\infty \mathbf{1}_{\{\|U^n_t\|_{\mathbb{L}^2_\nu}>n\}}\big\|U^n_t\big\|^{\frac{2+\varpi}{1+\varpi}}_{\mathbb{L}^2_\nu}dt\right)^{\frac{1+\varpi}{2+\varpi}}\right] \\
&\le \frac{2}{n}C^1_{\widetilde\Lambda}+C^{2+\varpi}_{\widetilde\Lambda}(n-1)^{-\frac{\varpi}{2+\varpi}}\Big(\big\|Z^n\big\|^{\frac{2+2\varpi}{2+\varpi}}_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{R}^{l\times d})}+\big\|U^n\big\|^{\frac{2+2\varpi}{2+\varpi}}_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{L}^2_\nu)}\Big) \\
&\le \frac{2}{n}C^1_{\widetilde\Lambda}+2\,C^{2+\varpi}_{\widetilde\Lambda}(n-1)^{-\frac{\varpi}{2+\varpi}}(R-2)^{\frac{2+2\varpi}{2+\varpi}},
\end{aligned}$$

where $C^1_{\widetilde{\Lambda}} \stackrel{\triangle}{=} E\int_0^\infty \widetilde{\Lambda}_t dt$ and $C^{2+\varpi}_{\widetilde{\Lambda}} \stackrel{\triangle}{=} \Big\{E\int_0^\infty \big(\widetilde{\Lambda}_t\big)^{2+\varpi} dt\Big\}^{\frac{1}{2+\varpi}}$. In the last inequality, we applied Lemma A.1 with $q=\frac{1+\varpi}{2+\varpi}$, $n=2$, $a_1=\big\|Z^n\big\|^2_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{R}^{l\times d})}$ and $a_2=\big\|U^n\big\|^2_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{L}^2_\nu)}$.

*3.*
$$
\begin{aligned}
E\int_0^\infty \widetilde{I}^2_t dt &\le (1+2^{p-1})n^{1-p}\bigg(2R\bigg\|\int_0^\infty \beta_t dt\bigg\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}+\int_0^\infty 2\big[c(t)+c^2(t)\big]dt\\
&\quad+\frac12\Big(\big\|Z^m\big\|^2_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{R}^{l\times d})}+\big\|Z^n\big\|^2_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{R}^{l\times d})}+\big\|U^m\big\|^2_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{L}^2_\nu)}+\big\|U^n\big\|^2_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{L}^2_\nu)}\Big)\bigg)\\
&\le (1+2^{p-1})n^{1-p}\bigg(2R\bigg\|\int_0^\infty \beta_t dt\bigg\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}+\int_0^\infty 2\big[c(t)+c^2(t)\big]dt+2(R-2)^2\bigg).
\end{aligned}
$$

*4.*
$$
\begin{aligned}
E\int_0^\infty \widetilde{I}^3_t dt &\le (2R-4)^{p-1}\bigg(E\int_0^\infty (1-\beta^n_t)\big(R\beta_t+c(t)\big)dt+\big(\|Z^n\|_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{R}^{l\times d})}+\|U^n\|_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{L}^2_\nu)}\big)\Big\{E\int_0^\infty c^2(t)(1-\beta^n_t)^2dt\Big\}^{\frac12}\bigg)\\
&\le (2R-4)^{p-1}\bigg(E\int_0^\infty (1-\beta^n_t)\big(R\beta_t+c(t)\big)dt+2(R-2)\Big\{E\int_0^\infty c^2(t)(1-\beta^n_t)^2dt\Big\}^{\frac12}\bigg)\stackrel{\triangle}{=} J_n.
\end{aligned}
$$

Because $\beta^n_t=\frac{e^{-t}}{e^{-t}\vee(\beta_t/n)}\nearrow 1$ as $n\to\infty$, $\forall\, t\in[0,\infty)$, the Dominated Convergence Theorem gives that $\lim\limits_{n\to\infty} J_n=0$. Thus, letting $n\to\infty$ in (2.57) verifies the condition (2.22). Moreover, since $\|\cdot\|_{\mathbb{D}^p_{\mathbf{F}}}\le\|\cdot\|_{\mathbb{D}^\infty_{\mathbf{F}}}$, $\|\cdot\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})}\le\|\cdot\|_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{R}^{l\times d})}$ and $\|\cdot\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{L}^2_\nu)}\le\|\cdot\|_{\mathbb{M}^2_{\mathbf{F}}(\mathbb{L}^2_\nu)}$ by Hölder's inequality, we see from (2.46) that (2.23) also holds. Then Proposition 2.2 shows that $\big\{(Y^n,Z^n,U^n)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{S}^p_{\mathbf{F}}$. Let $(Y,Z,U)$ be its limit in $\mathbb{S}^p_{\mathbf{F}}$.

(**4**) Since
$$
\lim_{n\to\infty}\big\|Y^n-Y\big\|^p_{\mathbb{D}^p_{\mathbf{F}}} = \lim_{n\to\infty}E\bigg[\sup_{t\in[0,\infty)}\big|Y^n_t-Y_t\big|^p\bigg]=0, \tag{2.58}
$$
$$
\lim_{n\to\infty}\big\|Z^n-Z\big\|^p_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})} = \lim_{n\to\infty}E\bigg[\Big(\int_0^\infty\big|Z^n_t-Z_t\big|^2dt\Big)^{\frac p2}\bigg]=0, \tag{2.59}
$$
$$
\lim_{n\to\infty}\big\|U^n-U\big\|^p_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{L}^2_\nu)}\text{and} = \lim_{n\to\infty}E\bigg[\Big(\int_0^\infty\big\|U^n_t-U_t\big\|^2_{\mathbb{L}^2_\nu}dt\Big)^{\frac p2}\bigg]=0, \tag{2.60}
$$

we can extract a subsequence $\{n_i\}_{i\in\mathbb{N}}$ from $\mathbb{N}$ such that
$$
\big\|Z^{n_{i+1}}-Z^{n_i}\big\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})}\vee\big\|U^{n_{i+1}}-U^{n_i}\big\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{L}^2_\nu)}\le 2^{-i},\qquad\forall\, i\in\mathbb{N}
$$
and that $P$-a.s.
$$
\lim_{i\to\infty}\sup_{t\in[0,\infty)}\big|Y^{n_i}_t-Y_t\big|^p=\lim_{i\to\infty}\Big(\int_0^\infty\big|Z^{n_i}_t-Z_t\big|^2dt\Big)^{\frac p2}=\lim_{i\to\infty}\Big(\int_0^\infty\big\|U^{n_i}_t-U_t\big\|^2_{\mathbb{L}^2_\nu}dt\Big)^{\frac p2}=0,
$$
thus
$$
\lim_{i\to\infty}\sup_{t\in[0,\infty)}\big|Y^{n_i}_t-Y_t\big|=\lim_{i\to\infty}\int_0^\infty\big|Z^{n_i}_t-Z_t\big|^2dt=\lim_{i\to\infty}\int_0^\infty\big\|U^{n_i}_t-U_t\big\|^2_{\mathbb{L}^2_\nu}dt=0. \tag{2.61}
$$
By (2.46), it holds $P$-a.s. that
$$
\sup_{t\in[0,\infty)}|Y_t|\le\sup_{t\in[0,\infty)}\big|Y_t-Y^{n_i}_t\big|+\sup_{t\in[0,\infty)}\big|Y^{n_i}_t\big|\le\sup_{t\in[0,\infty)}\big|Y^{n_i}_t-Y_t\big|+\big\|Y^{n_i}\big\|_{\mathbb{D}^\infty_{\mathbf{F}}}\le\sup_{t\in[0,\infty)}\big|Y^{n_i}_t-Y_t\big|+R-2,\quad\forall\, i\in\mathbb{N}.
$$
Letting $i\to\infty$, we see from (2.61) that $\sup\limits_{t\in[0,\infty)}|Y_t|\le R-2$, $P$-a.s., which implies that $\|Y\|_{\mathbb{D}^\infty_{\mathbf{F}}}\le R-2$.

For any $i\in\mathbb{N}$, we define two real-valued, $\mathbf{F}$-predictably measurable processes
$$
\mathcal{Z}^i_t\stackrel{\triangle}{=}|Z_t|+\sum_{j=1}^i\big|Z^{n_j}_t-Z^{n_{j-1}}_t\big|\quad\text{and}\quad\mathcal{U}^i_t\stackrel{\triangle}{=}\|U_t\|_{\mathbb{L}^2_\nu}+\sum_{j=1}^i\big\|U^{n_j}_t-U^{n_{j-1}}_t\big\|_{\mathbb{L}^2_\nu},\quad t\in[0,\infty)
$$

with $Z^{n_0} \stackrel{\triangle}{=} Z$ and $U^{n_0} \stackrel{\triangle}{=} U$. One can easily deduce that $\mathcal{Z}^i \in \mathbb{M}^p_{\mathbf{F}}(\mathbb{R})$ with

$$\big\|\mathcal{Z}^i\big\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R})} \le \big\|Z\big\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})} + \sum_{j=1}^{i} \big\|Z^{n_j} - Z^{n_{j-1}}\big\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})} \le 1 + \big\|Z\big\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})} + \big\|Z^{n_1} - Z\big\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})}. \tag{2.62}$$

As $\big\{\mathcal{Z}^i\big\}_{i\in\mathbb{N}}$ is an increasing sequence in $i$, we set $\mathcal{Z}_t \stackrel{\triangle}{=} \lim\limits_{i\to\infty}\uparrow \mathcal{Z}^i_t = |Z_t| + \sum\limits_{j=1}^{\infty}\big|Z^{n_j}_t - Z^{n_{j-1}}_t\big|$, $t \in [0,\infty)$. The Monotone Convergence Theorem shows that for any $\omega \in \Omega$

$$\int_0^\infty \big(\mathcal{Z}_t(\omega)\big)^2 dt = \lim_{i\to\infty}\uparrow \int_0^\infty \big(\mathcal{Z}^i_t(\omega)\big)^2 dt, \quad \text{thus} \quad \bigg(\int_0^\infty \big(\mathcal{Z}_t(\omega)\big)^2 dt\bigg)^{\frac{p}{2}} = \lim_{i\to\infty}\uparrow \bigg(\int_0^\infty \big(\mathcal{Z}^i_t(\omega)\big)^2 dt\bigg)^{\frac{p}{2}}.$$

Applying the Monotone Convergence Theorem once again, we can deduce from (2.62) and Lemma A.1 that

$$E\bigg[\bigg(\int_0^\infty \mathcal{Z}^2_t dt\bigg)^{\frac{p}{2}}\bigg] = \lim_{i\to\infty}\uparrow E\bigg[\bigg(\int_0^\infty \big(\mathcal{Z}^i_t\big)^2 dt\bigg)^{\frac{p}{2}}\bigg] \le 3^{p-1}\Big(1+\big\|Z\big\|^p_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})}+\big\|Z^{n_1}-Z\big\|^p_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})}\Big) < \infty, \tag{2.63}$$

which implies that

$$\bigg(\int_0^\infty \mathcal{Z}^2_t dt\bigg)^{\frac{p}{2}} < \infty, \quad \text{thus} \quad \int_0^\infty \mathcal{Z}^2_t dt < \infty, \quad P\text{-a.s.} \tag{2.64}$$

Similarly, the process $\mathcal{U}_t \stackrel{\triangle}{=} \lim\limits_{i\to\infty}\uparrow \mathcal{U}^i_t = \|U_t\|_{\mathbb{L}^2_\nu} + \sum\limits_{j=1}^{\infty}\big\|U^{n_j}_t - U^{n_{j-1}}_t\big\|_{\mathbb{L}^2_\nu}$, $t \in [0,\infty)$ satisfies

$$E\bigg[\bigg(\int_0^\infty \mathcal{U}^2_t dt\bigg)^{\frac{p}{2}}\bigg] < \infty, \quad \text{thus} \quad \int_0^\infty \mathcal{U}^2_t dt < \infty, \quad P\text{-a.s.} \tag{2.65}$$

Fix $k \in \mathbb{N}$. We define an $\mathbf{F}_\infty$-stopping time

$$\tau_k \stackrel{\triangle}{=} \inf\bigg\{t \in [0,\infty) : \int_0^t \big(\mathcal{Z}^2_s + \mathcal{U}^2_s\big) ds > k\bigg\}. \tag{2.66}$$

Since $\int_0^{\tau_k}\Big(\big|Z^{n_i}_t - Z_t\big|^2 + \big\|U^{n_i}_t - U_t\big\|^2_{\mathbb{L}^2_\nu}\Big) dt \le \int_0^{\tau_k}\Big(\big(\mathcal{Z}^i_t\big)^2 + \big(\mathcal{U}^i_t\big)^2\Big) dt \le \int_0^{\tau_k}\big(\mathcal{Z}^2_t + \mathcal{U}^2_t\big) dt \le k$ , $\forall\, \omega \in \Omega$, the Bounded Convergence Theorem and (2.61) show that

$$\lim_{i\to\infty} E\bigg[\int_0^{\tau_k}\big|Z^{n_i}_t - Z_t\big|^2 dt\bigg] = \lim_{i\to\infty} E\bigg[\int_0^{\tau_k}\big\|U^{n_i}_t - U_t\big\|^2_{\mathbb{L}^2_\nu} dt\bigg] = 0. \tag{2.67}$$

Hence, there exists a subsequence $\big\{n^k_i\big\}_{i\in\mathbb{N}}$ of $\big\{n_i\big\}_{i\in\mathbb{N}}$ such that for $dt \otimes dP$-a.e. $(t,\omega) \in [0,\infty)\times\Omega$

$$\lim_{i\to\infty} \mathbf{1}_{\{t\le\tau_k\}}\big|Z^{n^k_i}_t - Z_t\big|^2 = \lim_{i\to\infty} \mathbf{1}_{\{t\le\tau_k\}}\big\|U^{n^k_i}_t - U_t\big\|^2_{\mathbb{L}^2_\nu} = 0. \tag{2.68}$$

**(5)** Next, let us show that

$$\lim_{i\to\infty} E\bigg[\int_0^{\tau_k}\Big|f_{n^k_i}\Big(t, Y^{n^k_i}_t, Z^{n^k_i}_t, U^{n^k_i}_t\Big) - f\big(t, Y_t, Z_t, U_t\big)\Big| dt\bigg] = 0. \tag{2.69}$$

By (2.41) and (2.46), it holds for any $i \in \mathbb{N}$ that

$$\begin{aligned} & E\bigg[\int_0^{\tau_k}\Big|f_{n^k_i}\Big(t, Y^{n^k_i}_t, Z^{n^k_i}_t, U^{n^k_i}_t\Big) - f\big(t, Y_t, Z_t, U_t\big)\Big| dt\bigg] \\ & \quad \le E\bigg[\int_0^{\tau_k}\int_{|(\tilde y,\tilde z)|<1}\Big|\beta^{n^k_i}_t f\Big(t, Y^{n^k_i}_t - \tfrac{1}{n^k_i}\tilde y, \pi_{n^k_i}\Big(Z^{n^k_i}_t - \tfrac{1}{n^k_i}\tilde z\Big), \pi_{n^k_i}\Big(U^{n^k_i}_t\Big)\Big) - f\big(t, Y_t, Z_t, U_t\big)\Big| \rho(\tilde y,\tilde z) d\tilde y d\tilde z dt\bigg]. \end{aligned} \tag{2.70}$$

Except on a $dt \otimes dP$-null set $\widetilde{\mathcal{N}}_2$ of $[0,\infty) \times \Omega$, We may assume that

( i) (H2), (H3) and (2.68) hold;

(ii) $\lim\limits_{i\to\infty} \left|Y_t^{n_i^k} - Y_t\right| = 0$ (by (2.61));

(iii) $|Y_t| \le R-2$ and $\left|Y_t^{n_i^k}\right| \le R-2,\ \forall\, i \in \mathbb{N}$.

Fix $(t,\omega) \in \widetilde{\mathcal{N}}_2^c$ with $t \le \tau_k(\omega)$, and fix $(\tilde{y}, \tilde{z}) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}$ with $|(\tilde{y},\tilde{z})| < 1$. It follows from (2.68) that

$$\lim_{i\to\infty} \left|Z_t^{n_i^k} - Z_t\right| = \lim_{i\to\infty} \left\|U_t^{n_i^k} - U_t\right\|_{\mathbb{L}^2_\nu} = 0. \tag{2.71}$$

With help of Lemma A.5, one can estimate as follows:

(e1) $\left|Y_t^{n_i^k} - \frac{1}{n_i^k}\tilde{y} - Y_t\right| \le \frac{1}{n_i^k} + \left|Y_t^{n_i^k} - Y_t\right| \to 0$, as $i\to\infty$;

(e2) $\left|\pi_{n_i^k}\Big(Z_t^{n_i^k} - \frac{1}{n_i^k}\tilde{z}\Big) - Z_t\right| \le \left|\pi_{n_i^k}\Big(Z_t^{n_i^k} - \frac{1}{n_i^k}\tilde{z}\Big) - \pi_{n_i^k}(Z_t)\right| + \left|\pi_{n_i^k}(Z_t) - Z_t\right| \le \left|Z_t^{n_i^k} - \frac{1}{n_i^k}\tilde{z} - Z_t\right| + \left|\pi_{n_i^k}(Z_t) - Z_t\right| \le \frac{1}{n_i^k} + \left|Z_t^{n_i^k} - Z_t\right| + \left|\pi_{n_i^k}(Z_t) - Z_t\right| \to 0$, as $i\to\infty$;

(e3) $\left\|\pi_{n_i^k}\Big(U_t^{n_i^k}\Big) - U_t\right\|_{\mathbb{L}^2_\nu} \le \left\|\pi_{n_i^k}\Big(U_t^{n_i^k}\Big) - \pi_{n_i^k}(U_t)\right\|_{\mathbb{L}^2_\nu} + \left\|\pi_{n_i^k}(U_t) - U_t\right\|_{\mathbb{L}^2_\nu} \le \left\|U_t^{n_i^k} - U_t\right\|_{\mathbb{L}^2_\nu} + \left\|\pi_{n_i^k}(U_t) - U_t\right\|_{\mathbb{L}^2_\nu} \to 0$, as $i\to\infty$.

Since the mapping $f\big(t,\cdot,\cdot,U_t\big)$ is continuous by (H1) and since $\lim\limits_{i\to\infty}\uparrow \beta_t^{n_i^k} = 1$, we can deduce from (e1) and (e2) that

$$\lim_{i\to\infty} \beta_t^{n_i^k} f\Big(t, Y_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{y}, \pi_{n_i^k}\Big(Z_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{z}\Big), U_t\Big) = f(t, Y_t, Z_t, U_t). \tag{2.72}$$

Moreover, (H3) shows that

$$\left|\beta_t^{n_i^k} f\Big(t, Y_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{y}, \pi_{n_i^k}\Big(Z_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{z}\Big), \pi_{n_i^k}\Big(U_t^{n_i^k}\Big)\Big) - \beta_t^{n_i^k} f\Big(t, Y_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{y}, \pi_{n_i^k}\Big(Z_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{z}\Big), U_t\Big)\right| \le c(t)\left\|\pi_{n_i^k}\Big(U_t^{n_i^k}\Big) - U_t\right\|_{\mathbb{L}^2_\nu},$$

which together with (2.72) and (e3) implies that

$$\lim_{i\to\infty} \left|\beta_t^{n_i^k} f\Big(t, Y_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{y}, \pi_{n_i^k}\Big(Z_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{z}\Big), \pi_{n_i^k}\Big(U_t^{n_i^k}\Big)\Big) - f(t, Y_t, Z_t, U_t)\right| = 0. \tag{2.73}$$

For any $i \in \mathbb{N}$, since $|Z_t^{n_i}| \le \mathcal{Z}_t^i \le \mathcal{Z}_t$ and $\|U_t^{n_i}\|_{\mathbb{L}^2_\nu} \le \mathcal{U}_t^i \le \mathcal{U}_t$, one can deduce from (H2) and (H3) that

$$\begin{aligned}
&\left|\beta_t^{n_i^k} f\Big(t, Y_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{y}, \pi_{n_i^k}\Big(Z_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{z}\Big), \pi_{n_i^k}\Big(U_t^{n_i^k}\Big)\Big) - f(t, Y_t, Z_t, U_t)\right| \\
&\quad \le \left|f\Big(t, Y_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{y}, \pi_{n_i^k}\Big(Z_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{z}\Big), \pi_{n_i^k}\Big(U_t^{n_i^k}\Big)\Big)\right| + |f(t, Y_t, Z_t, U_t)| \\
&\quad \le \Big(2 + \left|Y_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{y}\right| + |Y_t|\Big)\beta_t + c(t)\left(\left|\pi_{n_i^k}\Big(Z_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{z}\Big)\right| + |Z_t| + \left\|\pi_{n_i^k}\Big(U_t^{n_i^k}\Big)\right\|_{\mathbb{L}^2_\nu} + \|U_t\|_{\mathbb{L}^2_\nu}\right) \\
&\quad \le (2R-1)\beta_t + c(t)\left(1 + \left|Z_t^{n_i^k}\right| + |Z_t| + \left\|U_t^{n_i^k}\right\|_{\mathbb{L}^2_\nu} + \|U_t\|_{\mathbb{L}^2_\nu}\right) \le (2R-1)\beta_t + c(t)\big(1 + \mathcal{Z}_t + |Z_t| + \mathcal{U}_t + \|U_t\|_{\mathbb{L}^2_\nu}\big) \triangleq h_t.
\end{aligned}$$

Applying Holder's inequality, we see from (2.63) and (2.65) that

$$\begin{aligned}
&E\int_0^{\tau_k}\int_{|(\tilde{y},\tilde{z})|<1} h_s\,\rho(\tilde{y},\tilde{z})\,d\tilde{y}\,d\tilde{z}\,ds = E\int_0^{\tau_k} h_s ds \le E\int_0^\infty h_s ds \\
&\quad \le C + E\left[\left(\int_0^\infty c^2(s)ds\right)^{\frac{1}{2}}\left\{\left(\int_0^\infty \mathcal{Z}_s^2 ds\right)^{\frac{1}{2}} + \left(\int_0^\infty |Z_s|^2 ds\right)^{\frac{1}{2}} + \left(\int_0^\infty \mathcal{U}_s^2 ds\right)^{\frac{1}{2}} + \left(\int_0^\infty \|U_s\|^2_{\mathbb{L}^2_\nu} ds\right)^{\frac{1}{2}}\right\}\right] \\
&\quad \le C + \left(\int_0^\infty c^2(s)ds\right)^{\frac{1}{2}}\Big(\|\mathcal{Z}\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R})} + \|Z\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})} + \|\mathcal{U}\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R})} + \|U\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{L}^2_\nu)}\Big) < \infty
\end{aligned} \tag{2.74}$$

with $C \stackrel{\triangle}{=} (2R-1)\big\|\int_0^\infty \beta_s ds\big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)} + \int_0^\infty c(s)ds$. Hence, the Dominated Convergence Theorem and (2.73) show that

$$\lim_{i\to\infty} E\left[\int_0^{\tau_k}\int_{|(\tilde{y},\tilde{z})|<1}\left|\beta_t^{n_i^k} f\Big(t, Y_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{y}, \pi_{n_i^k}\Big(Z_t^{n_i^k} - \tfrac{1}{n_i^k}\tilde{z}\Big), \pi_{n_i^k}\Big(U_t^{n_i^k}\Big)\Big) - f\big(t, Y_t, Z_t, U_t\big)\right| \rho(\tilde{y},\tilde{z})d\tilde{y}d\tilde{z}dt\right] = 0,$$

which together with (2.70) leads to (2.69).

**(6)** Since[7] $\int_{(t\wedge\tau_k,\tau_k]} = \int_{(0,\tau_k]} - \int_{(0,t\wedge\tau_k]}$, $\forall\, t \in [0,\infty)$. the Burkholder-Davis-Gundy inequality and Hölder's inequality imply that

$$\begin{aligned}
&E\left[\sup_{t\in[0,\infty)}\left|\int_{(t\wedge\tau_k,\tau_k]}\int_{\mathcal{X}}\big(U_s^{n_i^k}(x) - U_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)\right|\right] \le 2E\left[\sup_{t\in[0,\infty)}\left|\int_{(0,t\wedge\tau_k]}\int_{\mathcal{X}}\big(U_s^{n_i^k}(x) - U_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)\right|\right]\\
&\quad \le c_0\, E\left[\left\{\int_{(0,\tau_k]}\int_{\mathcal{X}}\big|(U_s^{n_i^k}(x) - U_s(x))\big|^2 N_{\mathfrak{p}}(ds,dx)\right\}^{\frac{1}{2}}\right] \le c_0\left\{E\left[\int_{(0,\tau_k]}\int_{\mathcal{X}}\big|(U_s^{n_i^k}(x) - U_s(x))\big|^2 N_{\mathfrak{p}}(ds,dx)\right]\right\}^{\frac{1}{2}}\\
&\quad = c_0\left\{E\left[\int_0^{\tau_k}\big\|U_t^{n_i^k} - U_t\big\|^2_{\mathbb{L}^2_\nu}dt\right]\right\}^{\frac{1}{2}} \to 0, \ \text{ as } i\to\infty, \qquad (2.75)
\end{aligned}$$

and that

$$\begin{aligned}
E\left[\sup_{t\in[0,\infty)}\left|\int_{t\wedge\tau_k}^{\tau_k}\big(Z_s^{n_i^k} - Z_s\big)dB_s\right|\right] &\le 2E\left[\sup_{t\in[0,\infty)}\left|\int_0^{t\wedge\tau_k}\big(Z_s^{n_i^k} - Z_s\big)dB_s\right|\right] \le c_0 E\left[\left(\int_0^{\tau_k}\big|Z_s^{n_i^k} - Z_s\big|^2 ds\right)^{\frac{1}{2}}\right]\\
&\le c_0\left\|Z^{n_i^k} - Z\right\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})} \to 0, \ \text{ as } i\to\infty. \qquad (2.76)
\end{aligned}$$

In light of (2.61), (2.69), (2.75) and (2.76), there exists a subsequence $\{\widetilde{n}_i^k\}_{i\in\mathbb{N}}$ of $\{n_i^k\}_{i\in\mathbb{N}}$ such that except on a $P$-null set $N_1^k$

$$\begin{aligned}
\lim_{i\to\infty}\bigg\{&\sup_{t\in[0,\infty)}\big|Y_t^{\widetilde{n}_i^k} - Y_t\big| + \sup_{t\in[0,\infty)}\left|\int_{(t\wedge\tau_k,\tau_k]}\int_{\mathcal{X}}\big(U_s^{\widetilde{n}_i^k}(x) - U_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)\right|\\
&+ \int_0^{\tau_k}\left|f_{\widetilde{n}_i^k}\Big(t, Y_t^{\widetilde{n}_i^k}, Z_t^{\widetilde{n}_i^k}, U_t^{\widetilde{n}_i^k}\Big) - f\big(t, Y_t, Z_t, U_t\big)\right|dt + \sup_{t\in[0,\infty)}\left|\int_{t\wedge\tau_k}^{\tau_k}\big(Z_s^{\widetilde{n}_i^k} - Z_s\big)dB_s\right|\bigg\} = 0.
\end{aligned}$$

Since $\Big(Y^{\widetilde{n}_i^k}, Z^{\widetilde{n}_i^k}, U^{\widetilde{n}_i^k}\Big)$ solves BSDEJ$\Big(\xi, f_{\widetilde{n}_i^k}\Big)$ for any $i\in\mathbb{N}$, it holds except on a $P$-null set $N_2^k$ that

$$\begin{aligned}
Y_{t\wedge\tau_k}^{\widetilde{n}_i^k} \ = \ & \mathbf{1}_{\{\tau_k<\infty\}}Y_{\tau_k}^{\widetilde{n}_i^k} + \mathbf{1}_{\{\tau_k=\infty\}}\xi + \int_{t\wedge\tau_k}^{\tau_k} f_{\widetilde{n}_i^k}\Big(s, Y_s^{\widetilde{n}_i^k}, Z_s^{\widetilde{n}_i^k}, U_s^{\widetilde{n}_i^k}\Big)ds - \int_{t\wedge\tau_k}^{\tau_k} Z_s^{\widetilde{n}_i^k}\, dB_s\\
& - \int_{(t\wedge\tau_k,\tau_k]}\int_{\mathcal{X}} U_s^{\widetilde{n}_i^k}(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \qquad t\in[0,\infty), \qquad \forall\, i\in\mathbb{N}. \qquad (2.77)
\end{aligned}$$

For any $\omega\in\Omega_k \stackrel{\triangle}{=} \big(N_1^k\big)^c \cap \big(N_2^k\big)^c$ and any $t\in[0,\infty)$, letting $i\to\infty$ in (2.77), we obtain that over $\Omega_k$

$$\begin{aligned}
Y_{t\wedge\tau_k} \ = \ & \mathbf{1}_{\{\tau_k<\infty\}}Y_{\tau_k} + \mathbf{1}_{\{\tau_k=\infty\}}\xi + \int_{t\wedge\tau_k}^{\tau_k} f\big(s, Y_s, Z_s, U_s\big)ds - \int_{t\wedge\tau_k}^{\tau_k} Z_s\, dB_s\\
& - \int_{(t\wedge\tau_k,\tau_k]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \qquad t\in[0,\infty). \qquad (2.78)
\end{aligned}$$

Thanks to (2.64) and (2.65), one can find a $P$-null set $N_3$ such that for any $\omega\in N_3^c$, $\tau_k(\omega)=\infty$ for some $k=k(\omega)\in\mathbb{N}$. Eventually, for any $\omega\in\widetilde{\Omega} \stackrel{\triangle}{=} \Big(\underset{k\in\mathbb{N}}{\cap}\Omega_k\Big)\cap N_3^c = \Big(\underset{k\in\mathbb{N}}{\cap}\big(N_1^k\big)^c\Big)\cap\Big(\underset{k\in\mathbb{N}}{\cap}\big(N_2^k\big)^c\Big)\cap N_3^c$ and any $t\in[0,\infty)$, letting $k\to\infty$ in (2.78) shows that (1.2) holds over $\widetilde{\Omega}$. To wit, $(Y,Z,U)$ is a solution of BSDEJ$(\xi,f)$. □

[7] $(t\wedge\tau_k,\tau_k]$ stands for $(t,\infty)$ and $(0,\tau_k]$ denotes $(0,\infty)$ when $\tau_k=\infty$.

**Proof of Theorem 2.1: (Uniqueness)** Suppose that $(Y,Z,U)\in\mathbb{S}^p_{\mathbf{F}}$ and $(Y',Z',U')\in\mathbb{S}^p_{\mathbf{F}}$ are two solutions of the BSDEJ$(\xi,f)$. For any $n\in\mathbb{N}$, we set

$$(\xi_n,f_n)\stackrel{\triangle}{=}(\xi,f)\quad\text{ and }\quad(Y^n,Z^n,U^n)\stackrel{\triangle}{=}\left\{\begin{array}{ll}(Y,Z,U) & \text{if } n \text{ is odd,}\\ (Y',Z',U') & \text{if } n \text{ is even.}\end{array}\right.$$

For any $m,n\in\mathbb{N}$ with $m>n$, (H4) shows that (2.21) holds with $\delta_n=0$ and $\eta^{m,n}\equiv 0$. Thus, it is easy to see that (2.22) and (2.23) are both satisfied. Then Proposition 2.2 shows that $\big\{(Y^n,Z^n,U^n)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{S}^p_{\mathbf{F}}$, which implies that $(Y,Z,U)=(Y',Z',U')$ in the sense that $\|Y-Y'\|_{\mathbb{D}^p_{\mathbf{F}}}=\|Z-Z'\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})}=\|U-U'\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{L}^2_\nu)}=0$.

**(Existence)** For any $n\in\mathbb{N}$, we define $\xi_n\stackrel{\triangle}{=}\pi_n(\xi)$. Thanks to Proposition 2.3, the BSDEJ$(\xi_n,f)$ has a solution $(Y^n,Z^n,U^n)\in\mathbb{D}^\infty_{\mathbf{F}}\times\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})\times\mathbb{M}^p_{\mathbf{F}}\big(\mathbb{L}^2_\nu\big)$. For any $m$, $n\in\mathbb{N}$ with $m>n$, (H4) shows that $dt\otimes dP$-a.e.

$$\begin{array}{l}|Y^{m,n}_s|^{p-1}\left\langle\mathscr{D}\big(Y^{m,n}_s\big),f(s,Y^m_s,Z^m_s,U^m_s)-f(s,Y^n_s,Z^n_s,U^n_s)\right\rangle\\ \quad\le\ \lambda(s)\theta(|Y^{m,n}_s|^p)+\Lambda_s|Y^{m,n}_s|^p+\widetilde{\Lambda}_s|Y^{m,n}_s|^{p-1}\left(|Z^{m,n}_s|+\|U^{m,n}_s\|_{\mathbb{L}^2_\nu}\right).\end{array}$$

Hence, (2.21) holds with $f_n\equiv f$, $\delta_n=0$ and $\eta^{m,n}\equiv 0$. Clearly, (2.22) is automatically satisfied.

Suppose that $\int_0^\infty\lambda(t)dt>0$. We can deduce from (H2) and (H3) that $dt\otimes dP$-a.e.

$$\begin{array}{rcl}\big\langle Y^n_t,f(t,Y^n_t,Z^n_t,U^n_t)\big\rangle&\le&|Y^n_t|\left(\big|f(t,Y^n_t,Z^n_t,0)\big|+\big|f(t,Y^n_t,Z^n_t,U^n_t)-f(t,Y^n_t,Z^n_t,0)\big|\right)\\ &\le&|Y^n_t|\Big[\big(1+|Y^n_t|\big)\beta_t+c(t)\big(|Z^n_t|+\|U^n_t\|_{\mathbb{L}^2_\nu}\big)\Big]\\ &\le&\beta_t|Y^n_t|+\left(\beta_t+\dfrac{2}{p-1}c^2(t)\right)|Y^n_t|^2+\dfrac{p-1}{4}\big(|Z^n_t|^2+\|U^n_t\|^2_{\mathbb{L}^2_\nu}\big).\end{array}$$

Thus (2.1) is satisfied with $\mathfrak{f}_t=\beta_t$, $a_t=\beta_t+\frac{2}{p-1}c^2(t)$, $\ell_p=\frac{p-1}{4}$. Since $E\Big[\sup\limits_{t\in[0,\infty)}\big(e^{A_t}|Y^n_t|\big)^p\Big]\le e^{p\kappa_A}\big\|Y^n\big\|^p_{\mathbb{D}^\infty_{\mathbf{F}}}<\infty$, with $\kappa_A\stackrel{\triangle}{=}\big\|A_\infty\big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}\le\big\|\int_0^\infty\beta_sds\big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}+\frac{2}{p-1}\int_0^\infty c^2(s)ds$, Proposition 2.1 gives that

$$\begin{array}{rcl}\|Y^n\|^p_{\mathbb{D}^p_{\mathbf{F}}}+\|Z^n\|^p_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})}+\|U^n\|^p_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{L}^2_\nu)}&\le&c_p\,E\left[e^{pA_\infty}|\xi_n|^p+\left(\displaystyle\int_0^\infty e^{A_s}\beta_sds\right)^p\right]\\ &\le&c_pe^{p\kappa_A}\left(E\big[|\xi|^p\big]+\left\|\displaystyle\int_0^\infty\beta_sds\right\|^p_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}\right)<\infty,\end{array}$$

which implies (2.23). Then Proposition 2.2 shows that $\big\{(Y^n,Z^n,U^n)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{S}^p_{\mathbf{F}}$. Let $(Y,Z,U)$ be its limit in $\mathbb{S}^p_{\mathbf{F}}$.

The rest of proof is similar to that of Proposition 2.3 (cf. part (4)-(6) therein). By (2.58)-(2.60), we can find a subsequence $\{n_i\}_{i\in\mathbb{N}}$ from $\mathbb{N}$ such that

$$\big\|Y^{n_{i+1}}-Y^{n_i}\big\|_{\mathbb{D}^p_{\mathbf{F}}}\vee\big\|Z^{n_{i+1}}-Z^{n_i}\big\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})}\vee\big\|U^{n_{i+1}}-U^{n_i}\big\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{L}^2_\nu)}\le 2^{-i},\quad\forall\, i\in\mathbb{N}$$

and that (2.61) holds $P$-a.s. For any $i\in\mathbb{N}$, we set

$$\eta_i\stackrel{\triangle}{=}\sup_{t\in[0,\infty)}|Y_t|+\sum_{j=1}^i\sup_{t\in[0,\infty)}\big|Y^{n_j}_t-Y^{n_{j-1}}_t\big|$$

with $Y^{n_0}\stackrel{\triangle}{=}Y$. One can easily deduce that $\eta_i\in\mathbb{L}^p_+(\mathcal{F}_\infty)$ with

$$\begin{array}{rcl}\big\|\eta_i\big\|_{\mathbb{L}^p_+(\mathcal{F}_\infty)}&\le&\Big\|\sup\limits_{t\in[0,\infty)}|Y_t|\Big\|_{\mathbb{L}^p(\mathcal{F}_\infty)}+\displaystyle\sum_{j=1}^i\Big\|\sup_{t\in[0,\infty)}\big|Y^{n_j}_t-Y^{n_{j-1}}_t\big|\Big\|_{\mathbb{L}^p(\mathcal{F}_\infty)}\\ &=&\big\|Y\big\|_{\mathbb{D}^p_{\mathbf{F}}}+\displaystyle\sum_{j=1}^i\big\|Y^{n_j}-Y^{n_{j-1}}\big\|_{\mathbb{D}^p_{\mathbf{F}}}\le 1+\big\|Y\big\|_{\mathbb{D}^p_{\mathbf{F}}}+\big\|Y^{n_1}-Y\big\|_{\mathbb{D}^p_{\mathbf{F}}}.\end{array}\qquad(2.79)$$

As $\{\eta_i\}_{i\in\mathbb{N}}$ is an increasing sequence in $i$, we set $\eta \stackrel{\triangle}{=} \lim\limits_{i\to\infty}\uparrow \eta_i = \sup\limits_{t\in[0,\infty)} |Y_t| + \sum\limits_{j=1}^{\infty} \sup\limits_{t\in[0,\infty)} \big|Y^{n_j}_t - Y^{n_{j-1}}_t\big|$. Applying the Monotone Convergence Theorem once again, we can deduce from (2.79) and Lemma A.1 that

$$\|\eta\|^p_{\mathbb{L}^p_+(\mathcal{F}_\infty)} = E\big[\eta^p\big] = \lim_{i\to\infty}\uparrow E\big[\eta^p_i\big] \le 3^{p-1}\Big(1+\|Y\|^p_{\mathbb{D}^p_{\mathbf{F}}} + \|Y^{n_1}-Y\|^p_{\mathbb{D}^p_{\mathbf{F}}}\Big) < \infty. \tag{2.80}$$

We have seen in the proof of Proposition 2.3 that the two real-valued, $\mathbf{F}$-predictably measurable processes

$$\mathcal{Z}_t \stackrel{\triangle}{=} |Z_t| + \sum_{j=1}^{\infty}\big|Z^{n_j}_t - Z^{n_{j-1}}_t\big| \quad \text{and} \quad \mathcal{U}_t \stackrel{\triangle}{=} \|U_t\|_{\mathbb{L}^2_\nu} + \sum_{j=1}^{\infty}\big\|U^{n_j}_t - U^{n_{j-1}}_t\big\|_{\mathbb{L}^2_\nu}, \quad t\in[0,\infty),$$

with $Z^{n_0} \stackrel{\triangle}{=} Z$ and $U^{n_0} \stackrel{\triangle}{=} U$, satisfy

$$E\bigg[\Big(\int_0^\infty \mathcal{Z}^2_t dt\Big)^{\frac{p}{2}}\bigg] + E\bigg[\Big(\int_0^\infty \mathcal{U}^2_t dt\Big)^{\frac{p}{2}}\bigg] < \infty. \tag{2.81}$$

Thus $\int_0^\infty \mathcal{Z}^2_t dt < \infty$ and $\int_0^\infty \mathcal{U}^2_t dt<\infty$, $P$-a.s.

Fix $k\in\mathbb{N}$. We still define the $\mathbf{F}_\infty$-stopping time $\tau_k$ as in (2.66). Thanks to (2.67), one can extract a subsequence $\{n^k_i\}_{i\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$ such that (2.68) holds for $dt\otimes dP$-a.e. $(t,\omega)\in[0,\infty)\times\Omega$. Except on a $dt\otimes dP$-null set $\widetilde{\mathcal{N}}$ of $[0,\infty)\times\Omega$, We may assume that (H2), (H3) and (2.68) hold, as well as that $\lim\limits_{i\to\infty}\Big|Y^{n^k_i}_t - Y_t\Big| = 0$ $\big($by (2.61)$\big)$. Fix $(t,\omega)\in\widetilde{\mathcal{N}}^c$ with $t\le\tau_k(\omega)$. We still have (2.71) by (2.68). Then the continuity of the mapping $f\big(t,\cdot,\cdot,U_t\big)$ shows that

$$\lim_{i\to\infty} f\Big(t, Y^{n^k_i}_t, Z^{n^k_i}_t, U_t\Big) = f(t,Y_t,Z_t,U_t). \tag{2.82}$$

Moreover, (H3) shows that

$$\Big|f\Big(t,Y^{n^k_i}_t, Z^{n^k_i}_t, U^{n^k_i}_t\Big) - f\Big(t, Y^{n^k_i}_t, Z^{n^k_i}_t, U_t\Big)\Big| \le c(t)\Big\|U^{n^k_i}_t - U_t\Big\|_{\mathbb{L}^2_\nu},$$

which together with (2.82) implies that

$$\lim_{i\to\infty}\Big|f\Big(t,Y^{n^k_i}_t, Z^{n^k_i}_t, U^{n^k_i}_t\Big) - f(t,Y_t,Z_t,U_t)\Big| = 0. \tag{2.83}$$

For any $i\in\mathbb{N}$, since

$$\sup_{t\in[0,\infty)}|Y^{n_i}_t| \le \sup_{t\in[0,\infty)}|Y_t| + \sum_{j=1}^{i}\sup_{t\in[0,\infty)}\big|Y^{n_j}_t - Y^{n_{j-1}}_t\big| = \eta_i \le \eta, \ \ |Z^{n_i}_t| \le \mathcal{Z}_t \ \text{ and } \ \|U^{n_i}_t\|_{\mathbb{L}^2_\nu} \le \mathcal{U}_t,$$

one can deduce from (H2) and (H3) that

$$\begin{aligned}\Big|f\Big(t,Y^{n^k_i}_t, Z^{n^k_i}_t, U^{n^k_i}_t\Big) - f(t,Y_t,Z_t,U_t)\Big| &\le \Big(2+\Big|Y^{n^k_i}_t\Big| + |Y_t|\Big)\beta_t + c(t)\Big(\Big|Z^{n^k_i}_t\Big| + |Z_t| + \Big\|U^{n^k_i}_t\Big\|_{\mathbb{L}^2_\nu} + \|U_t\|_{\mathbb{L}^2_\nu}\Big)\\ &\le \Big(2+\eta+\sup_{s\in[0,\infty)}|Y_s|\Big)\beta_t + c(t)\big(\mathcal{Z}_t + |Z_t| + \mathcal{U}_t + \|U_t\|_{\mathbb{L}^2_\nu}\big).\end{aligned}$$

Applying Holder's inequality, we see from (2.80) that

$$\begin{aligned} E\bigg[\int_0^{\tau_k}\Big(2+\eta+\sup_{s\in[0,\infty)}|Y_s|\Big)\beta_t dt\bigg] &\le E\bigg[\Big(2+\eta+\sup_{s\in[0,\infty)}|Y_s|\Big)\int_0^\infty \beta_t dt\bigg]\\ &\le \big(2+\|\eta\|_{\mathbb{L}^p_+(\mathcal{F}_\infty)} + \|Y\|_{\mathbb{D}^p_{\mathbf{F}}}\big)\Big\|\int_0^\infty \beta_s ds\Big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)} < \infty.\end{aligned}$$

Similar to (2.74), Holder's inequality and (2.81) yield that

$$E\left[\int_0^{\tau_k} c(t)\big(\mathcal{Z}_t+|Z_t|+\mathcal{U}_t+\|U_t\|_{\mathbb{L}^2_\nu}\big)dt\right] \le \left(\int_0^\infty c^2(s)ds\right)^{\frac{1}{2}}\Big(\|\mathcal{Z}\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R})}+\|Z\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})}+\|\mathcal{U}\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{R})}+\|U\|_{\mathbb{M}^p_{\mathbf{F}}(\mathbb{L}^2_\nu)}\Big) < \infty.$$

Therefore, the Dominated Convergence Theorem shows that

$$\lim_{i\to\infty} E\left[\int_0^{\tau_k}\left|f\Big(t, Y_t^{n_i^k}, Z_t^{n_i^k}, U_t^{n_i^k}\Big) - f\big(t, Y_t, Z_t, U_t\big)\right| dt\right] = 0. \tag{2.84}$$

The two limits (2.75) and (2.76) still hold. Then using the similar arguments to those that lead to (1.2) and using the fact that $\lim\limits_{n\to\infty}\xi_n=\xi$, we can conclude that $(Y,Z,U)$ is a solution of BSDEJ$(\xi, f)$. □

Now, let $\gamma$ be an $\mathbf{F}_\infty$-stopping time that may take the infinite value $\infty$. Thanks to Theorem 2.1, the BSDEJ with random time horizon $\gamma$ is also well-posed for any terminal condition $\xi \in \mathbb{L}^p(\mathcal{F}_\gamma)$ under hypotheses (H1)-(H4).

**Corollary 2.1.** *Let $(\xi, f)$ be a parameter pair such that $\xi \in \mathbb{L}^p(\mathcal{F}_\gamma)$. If the generator $f$ satisfies (H1) for each $(t,\omega) \in [\![0,\gamma]\!]$[8] and satisfies (H2)-(H4) for $dt\otimes dP$-a.e. $(t,\omega)\in[\![0,\gamma]\!]$, then the following BSDEJ*

$$Y_{t\wedge\gamma} = \xi + \int_{t\wedge\gamma}^{\gamma} f(s, Y_s, Z_s, U_s)ds - \int_{t\wedge\gamma}^{\gamma} Z_s\, dB_s - \int_{(t\wedge\gamma,\gamma]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad \forall\, t\in[0,\infty); \quad P\text{-a.s.} \tag{2.85}$$

*admits a unique solution* $\Big\{\big(Y_t(\omega), Z_t(\omega), U_t(\omega)\big)\Big\}_{(t,\omega)\in[\![0,\gamma]\!]}$ *such that* $\Big\{\Big(Y_{t\wedge\gamma}, \mathbf{1}_{\{t\le\gamma\}}Z_t, \mathbf{1}_{\{t\le\gamma\}}U_t\Big)\Big\}_{t\in[0,\infty)} \in \mathbb{S}^p_{\mathbf{F}}$.

**Proof:** One can check that

$$\widetilde{f}(t,\omega,y,z,u) \stackrel{\triangle}{=} \mathbf{1}_{\{t\le\gamma(\omega)\}} f(t,\omega,y,z,u), \quad \forall\,(t,\omega,y,z,u)\in[0,\infty)\times\Omega\times\mathbb{R}^l\times\mathbb{R}^{l\times d}\times\mathbb{L}^2_\nu$$

defines a $\mathscr{P}\times\mathscr{B}\big(\mathbb{R}^l\big)\times\mathscr{B}\big(\mathbb{R}^{l\times d}\big)\times\mathscr{B}\big(\mathbb{L}^2_\nu\big)/\mathscr{B}(\mathbb{R}^l)$-measurable function that satisfies (H1) for each $(t,\omega)\in[0,\infty)\times\Omega$ and satisfies (H2)-(H4) for $dt\otimes dP$-a.e. $(t,\omega)\in[0,\infty)\times\Omega$. Theorem 2.1 then shows that the BSDEJ$\big(\xi,\widetilde{f}\,\big)$ admits a unique solution $(Y,Z,U)\in\mathbb{S}^p_{\mathbf{F}}$. So it holds except on a $P$-null set $N_1$ that

$$Y_t = \xi + \int_t^\infty \widetilde{f}(s, Y_s, Z_s, U_s)ds - \int_t^\infty Z_s\, dB_s - \int_{(t,\infty)}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,\infty). \tag{2.86}$$

Fix $T\in(0,\infty)$ and $k\in\mathbb{N}$. We define an $\mathbf{F}_\infty$-stopping time

$$\tau_k \stackrel{\triangle}{=} \inf\left\{t\in[0,\infty) : \int_0^t \Big(|Z_s|^2 + \big\|U_s\big\|^2_{\mathbb{L}^2_\nu}\Big)\, ds > k\right\}. \tag{2.87}$$

By (2.86), it holds on $N_1^c$ that

$$\begin{aligned} Y_{\gamma\wedge T\wedge\tau_k} &= Y_{T\wedge\tau_k} + \int_{\gamma\wedge T\wedge\tau_k}^{T\wedge\tau_k} \widetilde{f}(s, Y_s, Z_s, U_s)ds - \int_{\gamma\wedge T\wedge\tau_k}^{T\wedge\tau_k} Z_s\, dB_s - \int_{(\gamma\wedge T\wedge\tau_k, T\wedge\tau_k]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) \\ &= Y_{T\wedge\tau_k} - \int_{\gamma\wedge T\wedge\tau_k}^{T\wedge\tau_k} Z_s\, dB_s - \int_{(\gamma\wedge T\wedge\tau_k, T\wedge\tau_k]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx). \end{aligned}$$

Taking conditional expectation $E\big[\,\cdot\,\big|\mathcal{F}_{\gamma\wedge T\wedge\tau_k}\big]$ and multiplying $\mathbf{1}_{\{\gamma\le T\wedge\tau_k\}}$ on both sides yield that

$$\mathbf{1}_{\{\gamma\le T\wedge\tau_k\}}Y_\gamma = \mathbf{1}_{\{\gamma\le T\wedge\tau_k\}}Y_{\gamma\wedge T\wedge\tau_k} = \mathbf{1}_{\{\gamma\le T\wedge\tau_k\}}E\big[Y_{T\wedge\tau_k}\big|\mathcal{F}_{\gamma\wedge T\wedge\tau_k}\big] = \mathbf{1}_{\{\gamma\le T\wedge\tau_k\}}E\big[Y_{T\wedge\tau_k}\big|\mathcal{F}_\gamma\big], \quad P\text{-a.s.} \tag{2.88}$$

As $\big(Z_s,U_s\big)\in\mathbb{M}^p_{\mathbf{F}}(\mathbb{R}^{l\times d})\times\mathbb{M}^p_{\mathbf{F}}\big(\mathbb{L}^2_\nu\big)$, we see that $\int_0^\infty\Big(|Z_s|^2+\big\|U_s\big\|^2_{\mathbb{L}^2_\nu}\Big)\,ds<\infty$, $P$-a.s. Thus for $P$-a.s. $\omega\in\Omega$, $\tau_k(\omega)=\infty$ for some $k=k(\omega)\in\mathbb{N}$, which implies that $\lim\limits_{k\to\infty}Y_{T\wedge\tau_k}=Y_T$, $P$-a.s. although the process $Y$ may not be

[8] the stochastic interval $[\![0,\gamma]\!]$ is defined by $\{(t,\omega)\in[0,\infty)\times\Omega : t\le\gamma(\omega)\}$.

left-continuous. Since $E\Big[\sup\limits_{t\in[0,\infty)}|Y_t|\Big] \le \|Y\|_{\mathbb{D}^p_{\mathbf{F}}} < \infty$ by Hölder's inequality, the Dominated Convergence Theorem and (1.3) imply that

$$\lim_{k\to\infty} E\big[Y_{T\wedge\tau_k}\big|\mathcal{F}_\gamma\big] = E\big[Y_T\big|\mathcal{F}_\gamma\big], \quad \text{and} \quad \lim_{T\to\infty} E\big[Y_T\big|\mathcal{F}_\gamma\big] = E\big[\xi\big|\mathcal{F}_\gamma\big] = \xi, \quad P\text{-a.s.}$$

It is clear that $\lim\limits_{k\to\infty}\uparrow \mathbf{1}_{\{\gamma\le T\wedge\tau_k\}} = \mathbf{1}_{\{\gamma\le T\}}$ and that $\lim\limits_{T\to\infty}\uparrow \mathbf{1}_{\{\gamma\le T\}} = \mathbf{1}_{\{\gamma<\infty\}}$. Thus, letting $k\to\infty$ and then letting $T\to\infty$ in (2.88) give that $\mathbf{1}_{\{\gamma<\infty\}}Y_\gamma = \mathbf{1}_{\{\gamma<\infty\}}\xi$, $P$-a.s., which together with (1.3) implies that $Y_\gamma=\xi$ holds except on a $P$-null set $N_2$. Let $N=N_1\cup N_2$. It then holds on $N^c$ that

$$\int_\gamma^\infty \widetilde{f}(s,Y_s,Z_s,U_s)ds - \int_\gamma^\infty Z_s\,dB_s - \int_{(\gamma,\infty)}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) = Y_\gamma - \xi = 0.$$

Therefore, one can deduce from (2.86) that on $N^c$

$$\begin{aligned} Y_{t\wedge\gamma} &= \xi + \int_{t\wedge\gamma}^\infty \widetilde{f}(s,Y_s,Z_s,U_s)ds - \int_{t\wedge\gamma}^\infty Z_s\,dB_s - \int_{(t\wedge\gamma,\infty)}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \xi + \int_{t\wedge\gamma}^\gamma f(s,Y_s,Z_s,U_s)ds - \int_{t\wedge\gamma}^\gamma Z_s\,dB_s - \int_{(t\wedge\gamma,\gamma]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,\infty), \end{aligned}$$

which shows that $\Big\{\big(Y_t(\omega),Z_t(\omega),U_t(\omega)\big)\Big\}_{(t,\omega)\in[\![0,\gamma]\!]}$ is a solution of (2.85). Moreover, since $(Y,Z,U)\in\mathbb{S}^p_{\mathbf{F}}$, we easily see that $\Big\{\Big(Y_{t\wedge\gamma},\mathbf{1}_{\{t\le\gamma\}}Z_t,\mathbf{1}_{\{t\le\gamma\}}U_t\Big)\Big\}_{t\in[0,\infty)}$ belongs to $\mathbb{S}^p_{\mathbf{F}}$ as well.

On the other hand, if $\Big\{\big(\widetilde{Y}_t(\omega),\widetilde{Z}_t(\omega),\widetilde{U}_t(\omega)\big)\Big\}_{(t,\omega)\in[\![0,\gamma]\!]}$ is another solution of (2.85) such that $\Big\{\Big(\widetilde{Y}_{t\wedge\gamma},\mathbf{1}_{\{t\le\gamma\}}\widetilde{Z}_t,$ $\mathbf{1}_{\{t\le\gamma\}}\widetilde{U}_t\Big)\Big\}_{t\in[0,\infty)}\in\mathbb{S}^p_{\mathbf{F}}$, then it holds $P$-a.s. that

$$\begin{aligned} \widetilde{Y}_{t\wedge\gamma} &= \xi + \int_{t\wedge\gamma}^\gamma f(s,\widetilde{Y}_s,\widetilde{Z}_s,\widetilde{U}_s)ds - \int_{t\wedge\gamma}^\gamma \widetilde{Z}_s\,dB_s - \int_{(t\wedge\gamma,\gamma]}\int_{\mathcal{X}} \widetilde{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \xi + \int_t^\infty \mathbf{1}_{\{s\le\gamma\}} f(s,\widetilde{Y}_s,\widetilde{Z}_s,\widetilde{U}_s)ds - \int_t^\infty \mathbf{1}_{\{s\le\gamma\}}\widetilde{Z}_s\,dB_s - \int_{(t,\infty)}\mathbf{1}_{\{s\le\gamma\}}\int_{\mathcal{X}} \widetilde{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \xi + \int_t^\infty \widetilde{f}(s,\widetilde{Y}_{s\wedge\gamma},\mathbf{1}_{\{s\le\gamma\}}\widetilde{Z}_s,\mathbf{1}_{\{s\le\gamma\}}\widetilde{U}_s)ds - \int_t^\infty \mathbf{1}_{\{s\le\gamma\}}\widetilde{Z}_s\,dB_s - \int_{(t,\infty)}\int_{\mathcal{X}}\mathbf{1}_{\{s\le\gamma\}}\widetilde{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\ t\in[0,\infty), \end{aligned}$$

which shows that $\Big\{\Big(\widetilde{Y}_{t\wedge\gamma},\mathbf{1}_{\{t\le\gamma\}}\widetilde{Z}_t,\mathbf{1}_{\{t\le\gamma\}}\widetilde{U}_t\Big)\Big\}_{t\in[0,\infty)}$ also solves BSDEJ$\big(\xi,\widetilde{f}\big)$. Hence, the uniqueness of the solution of BSDEJ$\big(\xi,\widetilde{f}\big)$ in $\mathbb{S}^p_{\mathbf{F}}$ entails that

$$\begin{aligned} 0 &= E\Big[\sup_{t\in[0,\infty)}\big|Y_t-\widetilde{Y}_{t\wedge\gamma}\big|^p\Big] + E\bigg[\Big(\int_0^\infty\big|Z_t-\mathbf{1}_{\{t\le\gamma\}}\widetilde{Z}_t\big|^2dt\Big)^{\frac{p}{2}}\bigg] + E\bigg[\Big(\int_0^\infty\big\|U_t-\mathbf{1}_{\{t\le\gamma\}}\widetilde{U}_t\big\|^2_{\mathbb{L}^2_\nu}dt\Big)^{\frac{p}{2}}\bigg]\\ &\ge E\Big[\sup_{t\in[0,\gamma]}\big|Y_t-\widetilde{Y}_t\big|^p\Big] + E\bigg[\Big(\int_0^\gamma\big|Z_t-\widetilde{Z}_t\big|^2dt\Big)^{\frac{p}{2}}\bigg] + E\bigg[\Big(\int_0^\gamma\big\|U_t-\widetilde{U}_t\big\|^2_{\mathbb{L}^2_\nu}dt\Big)^{\frac{p}{2}}\bigg], \end{aligned}$$

which implies that $\Big\{\big(Y_t(\omega),Z_t(\omega),U_t(\omega)\big)\Big\}_{(t,\omega)\in[\![0,\gamma]\!]}$ is the unique solution of (2.85) such that $\Big\{\Big(Y_{t\wedge\gamma},\mathbf{1}_{\{t\le\gamma\}}Z_t,$ $\mathbf{1}_{\{t\le\gamma\}}U_t\Big)\Big\}_{t\in[0,\infty)}\in\mathbb{S}^p_{\mathbf{F}}$. □

# 3 Case 2: $p\in(2,\infty)$

**Theorem 3.1.** *Let $(\xi,f)$ be a parameter pair such that $\xi\in\mathbb{L}^p(\mathcal{F}_\infty)$ and that (H1) holds for each $(t,\omega)\in[0,\infty)\times\Omega$. Then the BSDEJ$(\xi,f)$ admits a unique solution $(Y,Z,U)\in\mathbb{D}^p_{\mathbf{F}}\times\mathbb{M}^2_{\mathbf{F}}(\mathbb{R}^{l\times d})\times\mathbb{M}^2_{\mathbf{F}}(\mathbb{L}^2_\nu)$ if the generator $f$ satisfies (H2), (H3) as well as the following condition (H4') for $dt\otimes dP$-a.e. $(t,\omega)\in[0,\infty)\times\Omega$:*

**(H4')** $\langle y_1-y_2, f\big(t,\omega,y_1,z_1,u_1\big)-f\big(t,y_2,z_2,u_2\big)\rangle,$
$$\le \Lambda_t|y_1-y_2|^2+\widetilde{\Lambda}_t|y_1-y_2|\big(|z_1-z_2|+\|u_1-u_2\|_{\mathbb{L}^2_\nu}\big), \quad \forall\,(y_1,z_1,u_1),(y_2,z_2,u_2)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}\times\mathbb{L}^2_\nu,$$
*where $\Lambda$ and $\widetilde{\Lambda}$ are two non-negative $\mathbf{F}$-progressively measurable processes defined in (h2).*

**Proof:** As $\xi\in\mathbb{L}^p(\mathcal{F}_\infty)\subset\mathbb{L}^2(\mathcal{F}_\infty)$, applying Theorem 2.1 with $p=2$ and $\lambda(\cdot)\equiv 0$, we know that the BSDEJ$(\xi,f)$ admits a unique solution $(Y,Z,U)\in\mathbb{S}^2_{\mathbf{F}}$. So it suffices to show that $Y\in\mathbb{D}^p_{\mathbf{F}}\subset\mathbb{D}^2_{\mathbf{F}}$. More precisely, we see from the proof of Theorem 2.1 that $(Y,Z,U)$ is the limit of $\{(Y^n,Z^n,U^n)\}_{n\in\mathbb{N}}$ in $\mathbb{S}^2_{\mathbf{F}}$, where $(Y^n,Z^n,U^n)$ is the unique solution of the BSDEJ$(\xi_n,f)$ with $\xi_n\stackrel{\triangle}{=}\pi_n(\xi)$ such that $Y^n\in\mathbb{D}^\infty_{\mathbf{F}}$.

Let us define $a_t\stackrel{\triangle}{=}\Lambda_t+3^{p-1}\widetilde{\Lambda}^2_t$ and $A_t\stackrel{\triangle}{=}\int_0^t a_s ds$, $t\in[0,\infty)$. It easily follows from (h2) that $A_\infty\in\mathbb{L}^\infty_+(\mathcal{F}_\infty)$ with $\kappa_A\stackrel{\triangle}{=}\big\|A_\infty\big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}\le\big\|\int_0^\infty\Lambda_s\,ds\big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}+3^{p-1}\Big\|\int_0^\infty\widetilde{\Lambda}^2_s\,ds\Big\|_{\mathbb{L}^\infty_+(\mathcal{F}_\infty)}$. Fix $k,m,n\in\mathbb{N}$ with $m>n$. We still set $(Y^{m,n},Z^{m,n},U^{m,n})\stackrel{\triangle}{=}(Y^m-Y^n,Z^m-Z^n,U^m-U^n)$ and define the $\mathbf{F}_\infty$-stopping time $\tau_k=\tau_k^{m,n}$ as in (2.24). The function $|x|^p$, $x\in\mathbb{R}^l$ has derivatives:
$$D_i|x|^p=p|x|^{p-2}x^i,\ \ \forall\,i\in\{1,\cdots,l\},\ \text{and}\ D^2_{ij}|x|^p=p|x|^{p-2}\delta_{ij}+p(p-2)\mathbf{1}_{\{x\neq 0\}}|x|^{p-4}x^ix^j,\ \ \forall\,i,j\in\{1,\cdots,l\}. \quad (3.1)$$

Fix $0\le t<T<\infty$. Similar to (2.6), applying Itô's formula to $e^{pA_s}|Y^{m,n}_s|^p$ over the interval $[\,t\wedge\tau_k,T\wedge\tau_k]$ yields that
$$\begin{aligned}
&e^{pA_{t\wedge\tau_k}}\left|Y^{m,n}_{t\wedge\tau_k}\right|^p+\frac12\int_{t\wedge\tau_k}^{T\wedge\tau_k}e^{pA_s}\,\mathrm{trace}\left(Z^{m,n}_s\big(Z^{m,n}_s\big)^T D^2|Y^{m,n}_s|^p\right)ds\\
&+\sum_{s\in(t\wedge\tau_k,T\wedge\tau_k]}e^{pA_s}\Big\{|Y^{m,n}_s|^p-|Y^{m,n}_{s-}|^p-\big\langle D|Y^{m,n}_{s-}|^p,\Delta Y^{m,n}_s\big\rangle\Big\}\\
&\quad= e^{pA_{T\wedge\tau_k}}\left|Y^{m,n}_{T\wedge\tau_k}\right|^p+p\int_{t\wedge\tau_k}^{T\wedge\tau_k}e^{pA_s}\left[|Y^{m,n}_s|^{p-2}\big\langle Y^{m,n}_s,f(s,Y^m_s,Z^m_s,U^m_s)-f(s,Y^n_s,Z^n_s,U^n_s)\big\rangle-a_s|Y^{m,n}_s|^p\right]ds\\
&\qquad-p\left(M^{m,n}_T-M^{m,n}_t+\widetilde{M}^{m,n}_T-\widetilde{M}^{m,n}_t\right),\qquad P\text{-a.s.}
\end{aligned}\quad (3.2)$$
where
$$\begin{aligned}
M^{m,n}_r&\stackrel{\triangle}{=}\int_0^{r\wedge\tau_k}e^{pA_s}|Y^{m,n}_s|^{p-2}\langle Y^{m,n}_s,Z^{m,n}_s\,dB_s\rangle\\
\text{and}\quad \widetilde{M}^{m,n}_r&\stackrel{\triangle}{=}\int_{(0,r\wedge\tau_k]}\int_{\mathcal{X}}e^{pA_s}|Y^{m,n}_{s-}|^{p-2}\langle Y^{m,n}_{s-},U^{m,n}_s(x)\rangle\widetilde{N}_{\mathfrak{p}}(ds,dx)
\end{aligned}$$
for any $r\in[0,\infty)$. It follows from (3.1) that
$$\begin{aligned}
\mathrm{trace}\left(Z^{m,n}_s\big(Z^{m,n}_s\big)^TD^2|Y^{m,n}_s|^p\right)&=p|Y^{m,n}_s|^{p-2}|Z^{m,n}_s|^2+p(p-2)\mathbf{1}_{\{Y^{m,n}_s\neq 0\}}|Y^{m,n}_s|^{p-4}\sum_{i=1}^l\bigg(\sum_{j=1}^d\big(Y^{m,n}_s\big)^i\big(Z^{m,n}_s\big)^{ij}\bigg)^2\\
&\ge p\,|Y^{m,n}_s|^{p-2}|Z^{m,n}_s|^2.
\end{aligned}\quad (3.3)$$
On the other hand, Taylor's Expansion Theorem and lemma A.4 imply that
$$\begin{aligned}
&\sum_{s\in(t\wedge\tau_k,T\wedge\tau_k]}e^{pA_s}\Big\{|Y^{m,n}_s|^p-|Y^{m,n}_{s-}|^p-\big\langle D|Y^{m,n}_{s-}|^p,\Delta Y^{m,n}_s\big\rangle\Big\}\\
&\quad= p\sum_{s\in(t\wedge\tau_k,T\wedge\tau_k]}e^{pA_s}\int_0^1(1-\alpha)\langle\Delta Y^{m,n}_s,D^2\big|Y^{m,n,\alpha}_s\big|^p\Delta Y^{m,n}_s\rangle d\alpha\quad\Big(\text{let }\ Y^{m,n,\alpha}_s\stackrel{\triangle}{=}Y^{m,n}_{s-}+\alpha\Delta Y^{m,n}_s\Big)\\
&\quad= p\sum_{s\in(t\wedge\tau_k,T\wedge\tau_k]}e^{pA_s}\int_0^1(1-\alpha)\Big[\big|Y^{m,n,\alpha}_s\big|^{p-2}|\Delta Y^{m,n}_s|^2+(p-2)\mathbf{1}_{\{Y^{m,n,\alpha}_s\neq0\}}\big|Y^{m,n,\alpha}_s\big|^{p-4}\big\langle\Delta Y^{m,n}_s,Y^{m,n,\alpha}_s\big\rangle^2\Big]d\alpha\\
&\quad\ge p\sum_{s\in(t\wedge\tau_k,T\wedge\tau_k]}e^{pA_s}|\Delta Y^{m,n}_s|^2\int_0^1(1-\alpha)\big|Y^{m,n,\alpha}_s\big|^{p-2}d\alpha\ge p\,3^{1-p}\sum_{s\in(t\wedge\tau_k,T\wedge\tau_k]}e^{pA_s}|\Delta Y^{m,n}_s|^2|Y^{m,n}_{s-}|^{p-2}.\\
&\quad= p\,3^{1-p}\int_{(t\wedge\tau_k,T\wedge\tau_k]}\int_{\mathcal{X}}e^{pA_s}|Y^{m,n}_{s-}|^{p-2}\big|U^{m,n}_s(x)\big|^2N_{\mathfrak{p}}(ds,dx).
\end{aligned}\quad (3.4)$$

Moreover, (H4') shows that $dt\otimes dP$-a.e.

$$\begin{aligned}\big\langle Y^{m,n}_s, f(s,Y^m_s,Z^m_s,U^m_s)-f(s,Y^n_s,Z^n_s,U^n_s)\big\rangle &\le \Lambda_s\big|Y^{m,n}_s\big|^2+\widetilde{\Lambda}_s\big|Y^{m,n}_s\big|\Big(\big|Z^{m,n}_s\big|+\big\|U^{m,n}_s\big\|_{\mathbb{L}^2_\nu}\Big)\\ &\le a_s\big|Y^{m,n}_s\big|^2+\frac{3^{1-p}}{2}\Big(\big|Z^{m,n}_s\big|^2+\big\|U^{m,n}_s\big\|^2_{\mathbb{L}^2_\nu}\Big). \end{aligned}\tag{3.5}$$

Since all processes in (3.2) are RCLL ones, Plugging (3.3)-(3.5) into (3.2) yields that $P$-a.s.

$$\begin{aligned}&e^{pA_{t\wedge\tau_k}}\,\big|Y^{m,n}_{t\wedge\tau_k}\big|^p+\frac{p3^{1-p}}{2}\int_{t\wedge\tau_k}^{T\wedge\tau_k}e^{pA_s}|Y^{m,n}_s|^{p-2}|Z^{m,n}_s|^2ds+p3^{1-p}\int_{(t\wedge\tau_k,T\wedge\tau_k]}\int_{\mathcal{X}}e^{pA_s}|Y^{m,n}_{s-}|^{p-2}|U^{m,n}_s(x)|^2N_{\mathfrak{p}}(ds,dx)\\ &\quad\le e^{pA_{T\wedge\tau_k}}\,\big|Y^{m,n}_{T\wedge\tau_k}\big|^p+\frac{p3^{1-p}}{2}\int_{t\wedge\tau_k}^{T\wedge\tau_k}e^{pA_s}|Y^{m,n}_s|^{p-2}\big\|U^{m,n}_s\big\|^2_{\mathbb{L}^2_\nu}ds\\ &\quad\quad -p\Big(M^{m,n}_T-M^{m,n}_t+\widetilde{M}^{m,n}_T-\widetilde{M}^{m,n}_t\Big),\quad t\in[0,T],\end{aligned}\tag{3.6}$$

where we used the fact that $\frac{1}{2}>3^{1-p}$.

The Burkholder-Davis-Gundy inequality and (2.24) imply that

$$\begin{aligned}E\bigg[\sup_{s\in[0,T]}\big|M^{m,n}_s\big|^2+\sup_{s\in[0,T]}\big|\widetilde{M}^{m,n}_s\big|^2\bigg] &\le c_0E\bigg[\int_0^{T\wedge\tau_k}e^{2pA_s}\Big(|Y^{m,n}_s|^{2p-2}|Z^{m,n}_s|^2+|Y^{m,n}_{s-}|^{2p-2}\big\|U^{m,n}_s\big\|^2_{\mathbb{L}^2_\nu}\Big)\,ds\bigg]\\ &\le c_0k\,e^{2p\kappa_A}\big\|Y^{m,n}\big\|^{2p-2}_{\mathbb{D}^\infty_{\mathbf{F}}}<\infty,\end{aligned}$$

which shows that both $M^{m,n}_{\cdot\wedge T}$ and $\widetilde{M}^{m,n}_{\cdot\wedge T}$ are uniformly integrable martingales. Similar to (2.11), one can deduce from (2.10) that

$$E\int_{(0,T\wedge\tau_k]}\int_{\mathcal{X}}e^{pA_s}|Y^{m,n}_{s-}|^{p-2}|U^{m,n}_s(x)|^2N_{\mathfrak{p}}(ds,dx)=E\int_0^{T\wedge\tau_k}e^{pA_s}|Y^{m,n}_s|^{p-2}\big\|U^{m,n}_s\big\|^2_{\mathbb{L}^2_\nu}ds.$$

Hence, letting $t=0$ in (3.6) and then taking expectations, we obtain

$$E\int_0^{T\wedge\tau_k}e^{pA_s}\,|Y^{m,n}_s|^{p-2}\Big(|Z^{m,n}_s|^2+\big\|U^{m,n}_s\big\|^2_{\mathbb{L}^2_\nu}\Big)\,ds\le\frac{2}{p}3^{p-1}e^{p\kappa_A}E\Big[\big|Y^{m,n}_{T\wedge\tau_k}\big|^p\Big].\tag{3.7}$$

It also follows from (3.6) that

$$\begin{aligned}&E\bigg[\sup_{t\in[0,T\wedge\tau_k]}\Big(e^{pA_t}|Y^{m,n}_t|^p\Big)\bigg]=E\bigg[\sup_{t\in[0,T]}\Big(e^{pA_{t\wedge\tau_k}}|Y^{m,n}_{t\wedge\tau_k}|^p\Big)\bigg]\\ &\quad\le e^{p\kappa_A}E\Big[\big|Y^{m,n}_{T\wedge\tau_k}\big|^p\Big]+\frac{p3^{1-p}}{2}E\int_0^{T\wedge\tau_k}e^{pA_s}|Y^{m,n}_s|^{p-2}\big\|U^{m,n}_s\big\|^2_{\mathbb{L}^2_\nu}ds+2pE\bigg[\sup_{t\in[0,T]}\big|M^{m,n}_t\big|+\sup_{t\in[0,T]}\big|\widetilde{M}^{m,n}_t\big|\bigg].\end{aligned}\tag{3.8}$$

Similar to (2.14), the Burkholder-Davis-Gundy inequality implies that

$$\begin{aligned}&2pE\bigg[\sup_{t\in[0,T]}\big|M^{m,n}_t\big|+\sup_{t\in[0,T]}\big|\widetilde{M}^{m,n}_t\big|\bigg]\\ &\quad\le c_pE\bigg[\bigg(\int_0^{T\wedge\tau_k}e^{2pA_s}|Y^{m,n}_s|^{2p-2}|Z^{m,n}_s|^2ds\bigg)^{\frac12}+\bigg(\int_{(0,T\wedge\tau_k]}\int_{\mathcal{X}}e^{2pA_s}|Y^{m,n}_{s-}|^{2p-2}|U^{m,n}_s(x)|^2N_{\mathfrak{p}}(ds,dx)\bigg)^{\frac12}\bigg]\\ &\quad\le\frac12E\bigg[\sup_{t\in[0,T\wedge\tau_k]}\Big(e^{pA_t}|Y^{m,n}_t|^p\Big)\bigg]+c_p\,E\int_0^{T\wedge\tau_k}e^{pA_s}|Y^{m,n}_s|^{p-2}\Big(|Z^{m,n}_s|^2+\big\|U^{m,n}_s\big\|^2_{\mathbb{L}^2_\nu}\Big)\,ds.\end{aligned}\tag{3.9}$$

Since $E\bigg[\sup_{t\in[0,T\wedge\tau_k]}\Big(e^{pA_t}|Y^{m,n}_t|^p\Big)\bigg]\le e^{p\kappa_A}\big\|Y^{m,n}\big\|^p_{\mathbb{D}^\infty_{\mathbf{F}}}<\infty$, putting (3.9) into (3.8), we can deduce from (3.7) that

$$E\bigg[\sup_{t\in[0,T\wedge\tau_k]}\big|Y^{m,n}_t\big|^p\bigg]\le E\bigg[\sup_{t\in[0,T\wedge\tau_k]}\Big(e^{pA_t}|Y^{m,n}_t|^p\Big)\bigg]\le c_p\,e^{p\kappa_A}E\Big[\big|Y^{m,n}_{T\wedge\tau_k}\big|^p\Big].\tag{3.10}$$

Since $\left(Z_s^{m,n}, U_s^{m,n}\right) \in \mathbb{M}_{\mathbf{F}}^2(\mathbb{R}^{l\times d}) \times \mathbb{M}_{\mathbf{F}}^2\left(\mathbb{L}_\nu^2\right)$, we see that $\int_0^\infty \left(|Z_s^{m,n}|^2 + \left\|U_s^{m,n}\right\|_{\mathbb{L}_\nu^2}^2\right) ds < \infty$, $P$-a.s. Thus for $P$-a.s. $\omega \in \Omega$, $\tau_k(\omega) = \infty$ for some $k = k(\omega) \in \mathbb{N}$, which leads to that $\lim\limits_{k\to\infty} Y_{T\wedge\tau_k}^{m,n} = Y_T^{m,n}$, $P$-a.s. although the process $Y^{m,n}$ may not be left-continuous. As $Y^{m,n} \in \mathbb{D}_{\mathbf{F}}^\infty$, the Bounded Convergence Theorem implies that

$$\lim_{k\to\infty} E\Big[\big|Y_{T\wedge\tau_k}^{m,n}\big|^p\Big] = E\Big[\big|Y_T^{m,n}\big|^p\Big] \quad \text{and} \quad \lim_{T\to\infty} E\Big[\big|Y_T^{m,n}\big|^p\Big] = E\Big[\big|\xi_m - \xi_n\big|^p\Big].$$

Therefore, letting $k \to \infty$ and then letting $T \to \infty$ in (3.10), we can deduce from the Monotone Convergence Theorem that

$$\left\|Y^{m,n}\right\|_{\mathbb{D}_{\mathbf{F}}^p}^p = E\left[\sup_{t\in[0,\infty)} \big|Y_t^{m,n}\big|^p\right] \le c_p\, e^{p\kappa_A} E\Big[\big|\xi_m - \xi_n\big|^p\Big] = c_p\, e^{p\kappa_A} \left\|\xi_m - \xi_n\right\|_{\mathbb{L}^p(\mathcal{F}_\infty)}^p. \tag{3.11}$$

Since $\lim\limits_{n\to\infty} \xi_n = \xi$, the Dominated Convergence Theorem implies that $\lim\limits_{n\to\infty} E\big[|\xi_n - \xi|^p\big] = 0$, i.e., $\xi_n$ converges to $\xi$ in $\mathbb{L}^p(\mathcal{F}_\infty)$. Hence, we see from (3.11) that $\left\{Y^n\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{D}_{\mathbf{F}}^p$. Let $\widetilde{Y}$ be its limit in $\mathbb{D}_{\mathbf{F}}^p$. As $\|\cdot\|_{\mathbb{D}_{\mathbf{F}}^2} \le \|\cdot\|_{\mathbb{D}_{\mathbf{F}}^p}$, $\left\{Y^n\right\}_{n\in\mathbb{N}}$ converges to $\widetilde{Y}$ also in $\mathbb{D}_{\mathbf{F}}^2$. Then the uniqueness of the limit of $\left\{Y^n\right\}_{n\in\mathbb{N}}$ in $\mathbb{D}_{\mathbf{F}}^2$ shows that $Y$ and $\widetilde{Y}$ are indistinguishable[9], which implies that $Y \in \mathbb{D}_{\mathbf{F}}^p$. $\square$

Similar to Corollary 2.1, we can deduce from Theorem 3.1 the following existence and uniqueness result of BSDEJ with random time horizon $\gamma$ for case "$p \in (2,\infty)$".

**Corollary 3.1.** *Let $(\xi, f)$ be a parameter pair such that $\xi \in \mathbb{L}^p(\mathcal{F}_\gamma)$. If the generator $f$ satisfies (H1) for each $(t,\omega) \in [\![0,\gamma]\!]$ and satisfies (H2), (H3) as well as (H4') for $dt \otimes dP$-a.e. $(t,\omega) \in [\![0,\gamma]\!]$, then the BSDEJ (2.85) admits a unique solution $\Big\{\big(Y_t(\omega), Z_t(\omega), U_t(\omega)\big)\Big\}_{(t,\omega)\in[\![0,\gamma]\!]}$ such that $\Big\{\Big(Y_{t\wedge\gamma}, \mathbf{1}_{\{t\le\gamma\}} Z_t, \mathbf{1}_{\{t\le\gamma\}} U_t\Big)\Big\}_{t\in[0,\infty)} \in \mathbb{D}_{\mathbf{F}}^p \times \mathbb{M}_{\mathbf{F}}^2(\mathbb{R}^{l\times d}) \times \mathbb{M}_{\mathbf{F}}^2(\mathbb{L}_\nu^2)$.*

# A Appendix

**Lemma A.1.** *Let $\{a_i\}_{i\in\mathbb{N}} \subset [0,\infty)$. For any $p \in (0,\infty)$ and $n \in \mathbb{N}$ with $n \ge 2$, we have*

$$\big(1 \wedge n^{p-1}\big) \sum_{i=1}^n a_i^p \le \left(\sum_{i=1}^n a_i\right)^p \le \big(1 \vee n^{p-1}\big) \sum_{i=1}^n a_i^p. \tag{A.1}$$

**Proof:** Suppose that $p \in (1,\infty)$ first. For any $0 \le b \le c < \infty$, one can deduce that

$$(b+c)^p - c^p = p\int_c^{b+c} t^{p-1} dt \ge p\int_c^{b+c} b^{p-1} dt = pb^p \ge b^p, \quad \text{or equivalent,} \quad (b+c)^p \ge b^p + c^p. \tag{A.2}$$

Thus, $(a_1+a_2)^p \ge a_1^p + a_2^p$. When $n \ge 3$, applying (A.2) consecutively, we obtain

$$\left(\sum_{i=1}^n a_i\right)^p \ge a_1^p + \left(\sum_{i=2}^n a_i\right)^p \ge a_1^p + a_2^p + \left(\sum_{i=3}^n a_i\right)^p \ge \cdots \ge \sum_{i=1}^{n-2} a_i^p + \left(\sum_{i=n-1}^n a_i\right)^p \ge \sum_{i=1}^n a_i^p. \tag{A.3}$$

Now, let $\mathfrak{m}_n$ be the counting probability measure on $S_n = \{1,\cdots,n\}$ with $\mathfrak{m}_n(i) = \frac{1}{n}$ for each $i \in S_n$. Jensen's Inequality implies that

$$\left(\sum_{i=1}^n \frac{a_i}{n}\right)^p = \left(\int_{S_n} a_i \mathfrak{m}_n(di)\right)^p \le \int_{S_n} a_i^p \mathfrak{m}_n(di) = \sum_{i=1}^n \frac{a_i^p}{n}.$$

Multiplying $n^p$ to both sides, we see from (A.3) that

$$\sum_{i=1}^n a_i^p \le \left(\sum_{i=1}^n a_i\right)^p \le n^{p-1} \sum_{i=1}^n a_i^p. \tag{A.4}$$

[9] i.e., $P\left(Y_t = \widetilde{Y}_t,\ \forall\, t \in [0,\infty)\right) = 1$.

Clearly, the case "$p=1$" is trivial. So it remains to show (A.1) for $p\in(0,1)$: Applying (A.4) with $\widetilde{p}=\frac{1}{p}$ and $\widetilde{a}_i=a_i^p$, $i\in S_n$ yields that

$$\sum_{i=1}^n a_i=\sum_{i=1}^n \widetilde{a}_i^{\widetilde{p}}\le\bigg(\sum_{i=1}^n \widetilde{a}_i\bigg)^{\widetilde{p}}=\bigg(\sum_{i=1}^n a_i^p\bigg)^{\frac{1}{p}}\le n^{\widetilde{p}-1}\sum_{i=1}^n \widetilde{a}_i^{\widetilde{p}}=n^{\frac{1}{p}-1}\sum_{i=1}^n a_i.$$

Taking $p$-th power on both inequalities above, we obtain

$$n^{p-1}\sum_{i=1}^n a_i^p\le\bigg(\sum_{i=1}^n a_i\bigg)^p\le\sum_{i=1}^n a_i^p. \qquad\square$$

**Lemma A.2.** *For any $b,c\in[0,\infty)$, we have*

$$\big|b^p-c^p\big|\le\begin{cases}|b-c|^p, & \text{if } p\in(0,1],\\ p(b\vee c)^{p-1}|b-c|, & \text{if } p\in(1,\infty).\end{cases}\tag{A.5}$$

**Proof:** It is trivial when $b=c$. Since $b$ and $c$ take the symmetric roles in (A.5), we only need to assume $b<c$ without loss of generality.

• When $p\in(0,1]$, applying Lemma A.1 with $a_1=b$ and $a_2=c-b$ yields that $c^p=(a_1+a_2)^p\le a_1^p+a_2^p=b^p+(c-b)^p$, which implies that $\big|b^p-c^p\big|=c^p-b^p\le(c-b)^p=|b-c|^p$;

• When $p\in(1,\infty)$, one can deduce that $c^p-b^p=p\int_b^c t^{p-1}dt\le p\int_b^c c^{p-1}dt=pc^{p-1}(c-b)$, which leads to that $\big|b^p-c^p\big|=c^p-b^p\le p\,c^{p-1}(c-b)=p(b\vee c)^{p-1}|b-c|$. $\square$

**Lemma A.3.** *Let $\theta,\zeta,\mu:[0,\infty)\mapsto[0,\infty)$ be three functions such that*

*( i) either $\theta\equiv 0$ or $\theta(t)>0$ for any $t>0$;*

*( ii) $\theta$ is increasing and satisfies $\int_{0+}^1\frac{1}{\theta(t)}dt=\infty$;*

*(iii) $\zeta$ is integrable and $\mu$ is bounded.*

*If $\mu(t)\le\int_t^\infty\theta\big(\mu(s)\big)\zeta(s)ds$ for any $t\ge 0$, then $\mu\equiv 0$.*

**Proof:** The case "$\theta\equiv 0$" is trivial. So we only assume that $\theta(t)>0$ for any $t>0$ by (i). It follows from (iii) that $\phi(0)\le\theta(\kappa_\mu)\int_0^\infty\zeta(s)ds<\infty$ with $\kappa_\mu\stackrel{\triangle}{=}\sup\limits_{s\in[0,\infty)}\mu(s)$. Thus

$$\phi(t)\stackrel{\triangle}{=}\int_t^\infty\theta\big(\mu(s)\big)\zeta(s)ds\in[0,\infty),\qquad\forall\,t\in[0,\infty).$$

defines a continuous and decreasing function. Since $\theta$ is increasing, differentiating function $\phi$ yields that

$$\phi'(t)=-\theta\big(\mu(t)\big)\zeta(t)\ge-\theta\big(\phi(t)\big)\zeta(t),\qquad\forall\,t\in[0,\infty).\tag{A.6}$$

Assume $\phi(0)>0$. Then $T\stackrel{\triangle}{=}\inf\{t\in(0,\infty):\phi(t)=0\}\in(0,\infty]$ and it is clear that $\lim\limits_{t\to T}\!\downarrow\phi(t)=0$. As the continuous and decreasing function $\phi$ images $[0,T)$ onto $(0,\phi(0)]$, Changing of variable gives that

$$\int_{0+}^{\phi(0)}\frac{1}{\theta(t)}dt=-\int_0^{T-}\frac{1}{\theta\big(\phi(t)\big)}d\phi(t)=-\int_0^{T-}\frac{\phi'(t)}{\theta\big(\phi(t)\big)}dt\le\int_0^{T-}\zeta(t)dt\le\int_0^\infty\zeta(t)dt<\infty,\tag{A.7}$$

where we used (A.6) and (iii). For any $0<a<b<\infty$, one can deduce from the monotonicity of function $\theta$ that $\int_a^b\frac{1}{\theta(t)}dt\le\int_a^b\frac{1}{\theta(a)}dt=\frac{b-a}{\theta(a)}<\infty$, which together with (A.7) implies that $\int_{0+}^1\frac{1}{\theta(t)}dt<\infty$. This results in a contradiction to assumption (ii). Therefore, $\phi(0)=0$, which forces $\phi(\cdot)\equiv 0$. As a consequence, $\mu(\cdot)\equiv 0$. $\square$

**Lemma A.4.** *Let $p\in(0,\infty)$ and let $\mathbb{B}$ be a generic real Banach space with norm $|\cdot|_{\mathbb{B}}$. For any $x,y\in\mathbb{B}$,*

$$\int_0^1(1-\alpha)|x+\alpha y|_{\mathbb{B}}^p d\alpha\ge 3^{-(1+p)}|x|_{\mathbb{B}}^p.$$

**Proof:** If $y=\mathbf{0}$, one simply has $\int_0^1(1-\alpha)|x|^p_{\mathbb{B}}d\alpha=\frac12|x|^p_{\mathbb{B}}$. So let us assume $y\neq\mathbf{0}$ and set $\alpha_0\stackrel{\triangle}{=}\frac{2|x|_{\mathbb{B}}}{3|y|_{\mathbb{B}}}$. Since it holds for any $\alpha\in[0,\alpha_0]\cup[2\alpha_0,\infty)$ that $\frac13|x|_{\mathbb{B}}\le\Big||x|_{\mathbb{B}}-\alpha|y|_{\mathbb{B}}\Big|\le|x+\alpha y|_{\mathbb{B}}$, we can discuss by three cases:

(1) When $1\le\alpha_0$ : $\int_0^1(1-\alpha)|x+\alpha y|^p_{\mathbb{B}}d\alpha\ge\big(\frac13|x|_{\mathbb{B}}\big)^p\int_0^1(1-\alpha)d\alpha=\frac12\big(\frac13|x|_{\mathbb{B}}\big)^p$;

(2) When $\frac12\le\alpha_0<1$ : $\int_0^1(1-\alpha)|x+\alpha y|^p_{\mathbb{B}}d\alpha\ge\int_0^{\frac12}(1-\alpha)|x+\alpha y|^p_{\mathbb{B}}d\alpha\ge\big(\frac13|x|_{\mathbb{B}}\big)^p\int_0^{\frac12}(1-\alpha)d\alpha=\frac38\big(\frac13|x|_{\mathbb{B}}\big)^p$;

(3) When $\alpha_0<\frac12$ : $\int_0^1(1-\alpha)|x+\alpha y|^p_{\mathbb{B}}d\alpha\ge\big(\frac13|x|_{\mathbb{B}}\big)^p\big\{\int_0^{\alpha_0}+\int_{2\alpha_0}^1\big\}(1-\alpha)d\alpha=\big(\frac13|x|_{\mathbb{B}}\big)^p\big(\frac32{\alpha_0}^2-\alpha_0+\frac12\big)\ge\frac13\big(\frac13|x|_{\mathbb{B}}\big)^p$. □

For the next two lemmas, we assume that $\mathbb{H}$ is a generic real Hilbert space with inner product $(\cdot,\cdot)_{\mathbb{H}}$.

**Lemma A.5.** *For any $x,y\in\mathbb{H}$, we have*

$$\big|\pi_r(x)-\pi_r(y)\big|_{\mathbb{H}}\le|x-y|_{\mathbb{H}},\qquad\forall\,r\in(0,\infty).\tag{A.8}$$

*Consequently,*

$$|x-y|_{\mathbb{H}}\ge\big(|x|_{\mathbb{H}}\wedge|y|_{\mathbb{H}}\big)|\mathscr{D}(x)-\mathscr{D}(y)|_{\mathbb{H}}.\tag{A.9}$$

**Proof:** Without loss of generality, we assume that $|x|_{\mathbb{H}}\le|y|_{\mathbb{H}}$. To see (A.8), let us discuss by three cases:

(1) When $r\ge|y|_{\mathbb{H}}$: Since $\pi_r(x)=x$ and $\pi_r(y)=y$, one simply has $\big|\pi_r(x)-\pi_r(y)\big|_{\mathbb{H}}=|x-y|_{\mathbb{H}}$;

(2) When $|x|_{\mathbb{H}}\le r<|y|_{\mathbb{H}}$: Let us set $\kappa\stackrel{\triangle}{=}\big(x,\mathscr{D}(y)\big)_{\mathbb{H}}$ and $\widehat{y}\stackrel{\triangle}{=}\kappa\mathscr{D}(y)$. Since $\big(x-\widehat{y},\mathscr{D}(y)\big)_{\mathbb{H}}=0$, it holds for any $\alpha\in\mathbb{R}$ that

$$\big|x-\alpha\mathscr{D}(y)\big|^2_{\mathbb{H}}=\big|x-\widehat{y}-(\alpha-\kappa)\mathscr{D}(y)\big|^2_{\mathbb{H}}=\big|x-\widehat{y}\big|^2_{\mathbb{H}}+\big|(\alpha-\kappa)\mathscr{D}(y)\big|^2_{\mathbb{H}}=\big|x-\widehat{y}\big|^2_{\mathbb{H}}+(\alpha-\kappa)^2.$$

Hence, it follows that

$$\big|\pi_r(x)-\pi_r(y)\big|^2_{\mathbb{H}}=\big|x-r\mathscr{D}(y)\big|^2_{\mathbb{H}}=\big|x-\widehat{y}\big|^2_{\mathbb{H}}+(r-\kappa)^2\le\big|x-\widehat{y}\big|^2_{\mathbb{H}}+\big(|y|_{\mathbb{H}}-\kappa\big)^2=\big|x-y\big|^2_{\mathbb{H}}.$$

where we used the fact that $\kappa\le\Big|\big(x,\mathscr{D}(y)\big)_{\mathbb{H}}\Big|\le|x|_{\mathbb{H}}\le r<|y|_{\mathbb{H}}$, thanks to the Schwarz inequality.

(3) When $r<|x|_{\mathbb{H}}$: We know from (2) that

$$|x-y|_{\mathbb{H}}\ge\big|\pi_{|x|_{\mathbb{H}}}(x)-\pi_{|x|_{\mathbb{H}}}(y)\big|_{\mathbb{H}}=\big|x-|x|_{\mathbb{H}}\mathscr{D}(y)\big|_{\mathbb{H}}=|x|_{\mathbb{H}}\big|\mathscr{D}(x)-\mathscr{D}(y)\big|_{\mathbb{H}}\ge r\big|\mathscr{D}(x)-\mathscr{D}(y)\big|_{\mathbb{H}}=\big|\pi_r(x)-\pi_r(y)\big|_{\mathbb{H}}.$$

If $x=\mathbf{0}$, (A.9) holds automatically. Otherwise, applying (A.8) with $r=|x|_{\mathbb{H}}$ gives rise to (A.9). □

**Lemma A.6.** *Let $p\in(0,1]$. For any $x,y\in\mathbb{H}$, we have* $\Big||x|^p_{\mathbb{H}}\mathscr{D}(x)-|y|^p_{\mathbb{H}}\mathscr{D}(y)\Big|_{\mathbb{H}}\le(1+2^p)|x-y|^p_{\mathbb{H}}$.

**Proof:** The case "$p=1$" is trivial since $\Big||x|_{\mathbb{H}}\mathscr{D}(x)-|y|_{\mathbb{H}}\mathscr{D}(y)\Big|_{\mathbb{H}}=|x-y|_{\mathbb{H}}$. For $p\in(0,1)$, we assume without loss of generality that $|x|_{\mathbb{H}}\le|y|_{\mathbb{H}}$ and discuss by three cases:

(1) When $x=0$: $\Big||y|^p_{\mathbb{H}}\mathscr{D}(y)\Big|_{\mathbb{H}}=|y|^p_{\mathbb{H}}$;

(2) When $0<|x|_{\mathbb{H}}\le|x-y|_{\mathbb{H}}$: $\Big||x|^p_{\mathbb{H}}\mathscr{D}(x)-|y|^p_{\mathbb{H}}\mathscr{D}(y)\Big|_{\mathbb{H}}\le\Big||x|^p_{\mathbb{H}}\mathscr{D}(x)\Big|_{\mathbb{H}}+\Big||y|^p_{\mathbb{H}}\mathscr{D}(y)\Big|_{\mathbb{H}}=|x|^p_{\mathbb{H}}+|y|^p_{\mathbb{H}}\le|x|^p_{\mathbb{H}}+(|x|_{\mathbb{H}}+|x-y|_{\mathbb{H}})^p\le(1+2^p)|x-y|^p_{\mathbb{H}}$;

(3) When $|x|_{\mathbb{H}}>|x-y|_{\mathbb{H}}$: $\Big||x|^p_{\mathbb{H}}\mathscr{D}(x)-|y|^p_{\mathbb{H}}\mathscr{D}(y)\Big|_{\mathbb{H}}\le|x|^p_{\mathbb{H}}\Big|\mathscr{D}(x)-\mathscr{D}(y)\Big|_{\mathbb{H}}+\Big||x|^p_{\mathbb{H}}-|y|^p_{\mathbb{H}}\Big|\le|x|^{p-1}_{\mathbb{H}}|x-y|_{\mathbb{H}}+\Big||x|_{\mathbb{H}}-|y|_{\mathbb{H}}\Big|^p<2|x-y|^p_{\mathbb{H}}$, where we used (A.9) and Lemma A.2 in the second inequality. □

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